(1) (24=10+4+10 points) Consider a sequence $h_0, h_1, h_2, \ldots$ given by the recurrence relation $h_n = 2h_{n-1} + 8h_{n-2}$ (for $n \geq 2$).

(a) Use the method of characteristic polynomial (also called auxiliary polynomial) to find the general solution (the one with constants c_1 and c_2).

Solution. The polynomial is $x^2 = 2x + 8$, which has the solutions $x = 4, -2$. Thus, the general solution is $h_n = c_1 4^n + c_2(-2)^n$.

(b) Use your general solution to find the specific solution that satisfies the initial conditions $h_0 = 1$ and $h_1 = 3$.

Solution. We set $h_0 = 1 = c_1 4^0 + c_2(-2)^0 = c_1 + c_2$ and $h_1 = 3 = c_1 4^1 + c_2(-2)^1$, i.e.,

$$c_1 + c_2 = 1 \text{ and } 4c_1 - 2c_2 = 3.$$

The values are $c_1 = \frac{5}{6}$ and $c_2 = \frac{1}{6}$, so the specific solution we want is $h_n = \frac{5}{6}4^n + \frac{1}{6}(-2)^n$.

(c) Then use your general solution in (a) to find the sequence that satisfies $h_n = 2h_{n-1} + 8h_{n-2} + 9n$ (for $n \geq 2$) and the initial conditions $h_0 = 1$ and $h_1 = 3$.

Solution. First step: Find a specific solution of this inhomogeneous recurrence. We try $h_n = an + b$. This gives

$$an + b = 2(a[n-1] + b) + 8(a[n-2] + b) + 9n,$$

which simplifies to

$$an + b = (2a + 8a + 9)n + (-18a + 10b).$$

Comparing coefficients of the two polynomials gives $a = 10a + 9$ and $b = -18a + 10b$, so $a = -1$ and $b = -2$. Thus, our special solution is $-n - 2$. Now we combine this with the general homogeneous solution from part (a) to get $h_n = c_1 4^n + c_2(-2)^n - n - 2$.

Second step: Fit this to the initial values: $h_0 = 1 = c_1 + c_2 - 2$ and $h_1 = 3 = -4c_1 + 2c_2 - 3$. The values are $c_1 = 2$ and $c_2 = 1$, so the specific solution we want is $h_n = 2(4)^n + (-2)^n - n - 2$.

(2) (6 points) What is the sequence h_n, $n = 0, 1, 2, \ldots$, whose generating function is $1/(1-4x)^{11}$?

Solution. By Newton's Binomial Theorem,

$$1/(1-4x)^{11} = \sum_{n=0}^{\infty} \binom{n + 11 - 1}{n} (4x)^n = \sum_{n=0}^{\infty} \binom{n+10}{10} 4^n x^n.$$

Thus, the sequence is $h_n = \binom{n+10}{10} 4^n$.

(3) (20 points) Use the method of generating functions to solve the recurrence $h_n = 5h_{n-1} + 3^n$ (for $n \geq 1$) with initial condition $h_0 = 1$.

Solution. First, set up the generating function (which I call $H(x)$ but you can call it whatever you please):

$$H(x) = \sum_{n=0}^{\infty} h_n x^n = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots . \tag{0.1}$$

(The dots are required because this is not a finite sum.)

Next, get the equation involving the generating function by summing x^n times the recurrence (for $n \geq 1$):

$$\sum_{n=1}^{\infty} h_n x^n = \sum_{n=1}^{\infty} 5h_{n-1} x^n + c. \tag{0.2}$$

We need to get this in terms of $H(x)$.

– The first sum is $h_1 x + h_2 x^2 + h_3 x^3 + \cdots = H(x) - h_0$.

– The second sum is $5x \sum_{n=1}^{\infty} h_{n-1} x^{n-1}$ (we matched the subscript to the exponent) $= \sum_{m=0}^{\infty} h_m x^m$ by the substitution $m = n - 1$. This gets the sum into the form $\sum h_m x^m$, as in $H(x)$, and since it begins at $m = 0$, this sum $= H(x)$.

– The third sum is $\sum_{n=1}^{\infty} (3x)^n = 3x \sum_{m=0}^{\infty} (3x)^m$ (by the substitution $m = n - 1$) $= \frac{3x}{1-3x}$ by the geometric series or Newton's Binomial Theorem.

Now we can put $H(x)$ into Equation (0.2):

$$H(x) - h_0 = 5xH(x) + \frac{3x}{1-3x}.$$

Solve for $H(x)$:

$$H(x)(1 - 5x) = h_0 + \frac{3x}{1-3x} = 1 + \frac{3x}{1-3x} = \frac{1}{1-3x}$$

and therefore

$$H(x) = \frac{1}{(1-5x)(1-3x)}.$$

Use partial fractions to get "pure" denominators:

$$\frac{1}{(1-5x)(1-3x)} = \frac{A}{1-5x} + \frac{B}{1-3x}$$

so $1 = A(1 - 3x) + B(1 - 5x)$, whose solution is $A = \frac{5}{2}$ and $B = -\frac{3}{2}$. Therefore,

$$H(x) = \frac{5}{2}\frac{1}{1-5x} - \frac{3}{2}\frac{1}{1-3x}.$$

Finally, we get the answer by applying Newton's Binomial Theorem (actually, simply the geometric series):

$$H(x) = \frac{5}{2}\sum_{n=0}^{\infty}(5x)^n - \frac{3}{2}\sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty}\left[\frac{5}{2}5^n x^n - \frac{3}{2}3^n x^n\right] = \sum_{n=0}^{\infty}\frac{5^{n+1} - 3^{n+1}}{2}x^n.$$

Comparing to (0.1) we see that the coefficient of x^n is h_n, so

$$h_n = \frac{5^{n+1} - 3^{n+1}}{2}.$$

Note: Optionally, you can check your answer by testing h_0 and h_1. The formula gives $h_0 = \frac{5^1-3^1}{2} = 1$, which is correct. It also gives $h_1 = \frac{5^2-3^2}{2} = 8$, which agrees with the recurrence, $h_1 = 5h_0 + 3^1 = 8$. That gives me confidence I did it right.

(4) (15 points) Miss Clavel takes 6 little girls to the little carousel, which has 6 little horses, each of a different color. The little girls get on the horses and enjoy a ride. The ride ends and they all get off for ice cream. Then they go back for another ride. In the second ride, no girl sits diametrically across from the same girl she did on the first ride. How many ways can the little girls sit in the second ride? (Sorry, no illustration; I can't match the original *Madeline*.)

Solution. The little horses are distinguishable, so we can number them from 1 to 6 around the carousel and let's number the girls also, according to which horse they sit on in the first ride. Notice that girls 1 and 4 are opposite, as are girls 2 and 5, and also girls 3 and 6. The opposite pairs each differ by 3.

In the second ride their seating is a permutation $s_1 s_2 \cdots s_6$ of $\{1, 2, \ldots, 6\}$, where s_1 is the number of the horse chosen by girl 1, etc. Counting directly gets messy so we apply the Principle of Inclusion and Exclusion ("apple PIE").

The universe U is the set of all possible seatings, i.e., all permutations of the girls, thus $|U| = 6!$. (We do not use circular permutations because the horses are of different colors.)

The properties are P_1: girls 1 and 4 are opposite; P_2: girls 2 and 5 are opposite; P_3: girls 3 and 6 are opposite. We want to avoid all three properties. With the usual definition of $A_i = \{x \in U : x \text{ satisfies } P_i\}$, we want to find $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|$. The PIE formula is

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |U| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|.$$

To find $|A_1|$ we notice that girl 1 has 6 choices of horse but then girl 4 must be opposite (1 choice). Then the remaining girls can sit in any way in the 4 unused horses (4! ways). Thus, $|A_1| = 6 \cdot 4!$. The computations for A_2 and A_3 are similar so $|A_2| = |A_3| = 6 \cdot 4!$.

To find $|A_1 \cap A_2|$ we let girl 1 choose from 6 horses; then girl 4 must be opposite. Next, girl 2 chooses from 4 horses and girl 5 must be opposite. Finally, girls 3 and 6 sit in the last two horses (2! choices). Thus, $|A_1 \cap A_2| = 6 \cdot 4 \cdot 2!$. Similarly, $|A_1 \cap A_3| = |A_2 \cap A_3| = 6 \cdot 4 \cdot 2!$.

Finally, if the girls satisfy all three properties, then girl 1 has 6 choices, girl 4 has one choice, girl 2 has 4 choices, girl 5 has one choice, girl 3 has 2 choices, and girl 6 has one choice. Thus, $|A_1 \cap A_2 \cap A_3| = 6 \cdot 4 \cdot 2$.

Substituting all this into the PIE formula,

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = 6! - 3(6 \cdot 4!) + 3(6 \cdot 4 \cdot 2) - 6 \cdot 4 \cdot 2.$$

This is the answer.

Note: If you calculated the actual number, it should be 384.

(5) (12 points) The Fibonacci numbers f_n, $n \geq 0$ satisfy the recurrence

$$f_{n+k-1} = f_k f_n + f_{k-1} f_{n-1} \text{ for all } n, k > 0.$$

Use this recurrence with $k = 7$ to prove that 8 divides f_n if and only if 6 divides n.

Solution. We need the values of f_6 and f_7, which are in the sequence

n	0	1	2	3	4	5	6	7
f_n	0	1	1	2	3	5	8	13

so $f_6 = 8$ and $f_7 = 13$. By the recurrence, $f_{n+6} = 8f_n + 13f_{n-1}$. Therefore,

$$8|f_{n+6} \iff 8|(13f_n + 8f_{n-1}) \iff 8|13f_n \iff 8|f_n$$

—the last step because $\gcd(8, 13) = 1$. This shows that f_n and f_{n+6} have the same property regarding divisibility by 8, namely, both are divisible or both are not divisible.

Therefore, all Fibonacci numbers $f_0, f_6, f_{12}, \ldots, f_{6j}, \ldots$ have the same divisibility property. Since $f_0 = 0$ (or f_6) is divisible by 8, all of them are divisible by 8. That is, $8|f_n$ if n is a multiple of 6. This proves half the statement.

Also, all of $f_1, f_7, f_{13}, \ldots, f_{6j+1}, \ldots$ have the same divisibility property. As $f_1 = 1$ is not divisible by 8, none of them is divisible by 8. Similarly,

- all f_{6j+2} have the same divisibility property as $f_2 = 1$ so they are not divisible by 8,
- all f_{6j+3} have the same divisibility property as $f_3 = 2$ so they are not divisible by 8,
- all f_{6j+4} have the same divisibility property as $f_4 = 3$ so they are not divisible by 8,
- all f_{6j+5} have the same divisibility property as $f_5 = 5$ so they are not divisible by 8.

This proves the other half of the statement.

One can write a formal proof of the last part with induction, but it is not quite simple and this is good enough.

A solution by induction on n is impossible.

(6) (10 points) Here is the Hasse diagram of a poset P. The table gives the Möbius function of P. There is an unknown function $F(x)$, $x \in Q$, and we define a function $G(y)$, $y \in Q$, by the formula

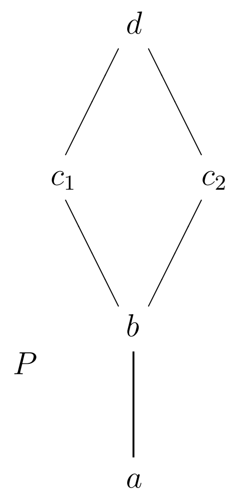
$$G(y) = \sum_{x:\, x \leq y} F(x).$$

The function G is known (see the table). Use Möbius inversion to find the value of $F(d)$.

x	a	b	c_1	c_2	d
$G(x)$	7	6	5	4	3

$\mu(x,y)$	a	b	c_1	c_2	d
a	1	-1	0	0	0
b	0	1	-1	-1	1
c_1	0	0	1	0	-1
c_2	0	0	0	1	-1
d	0	0	0	0	1

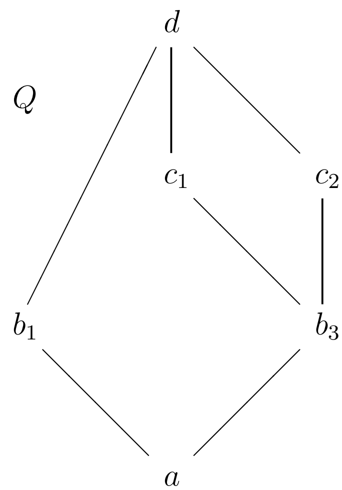
Values of $\mu(x,y)$: x on the left, y along the top.



Solution. The general formula for Möbius inversion is $F(y) = \sum_{x:\, x \leq y} \mu(x,y)G(x)$. Set $y = d$; then $F(d) = \sum_{x:\, x \leq d} \mu(x,d)G(x) = \sum_{x \in P} \mu(x,d)G(x)$ since every $x \in P$ is $\leq d$. So,

$$\begin{aligned} F(d) &= \mu(a,d)G(a) + \mu(b,d)G(b) + \mu(c_1,d)G(c_1) + \mu(c_2,d)G(c_2) + \mu(d,d)G(d) \\ &= (0)(7) + (1)(6) + (-1)(5) + (-1)(4) + (1)(3) \\ &= 0. \end{aligned}$$

(7) (13 points) Here is the Hasse diagram of a poset Q. Compute the Möbius function value $\mu(a, d)$.



Solution. We work upwards from a.

$\mu(a, a) = 1.$
$\mu(a, b_1) = -\mu(a, a) = -1.$
$\mu(a, b_3) = -\mu(a, a) = -1.$
$\mu(a, c_1) = -[\mu(a, a) + \mu(a, b_3)] = 0.$
$\mu(a, c_2) = -[\mu(a, a) + \mu(a, b_3)] = 0.$
$\mu(a, d) = -[\mu(a, a) + \mu(a, b_1) + \mu(a, b_3) + \mu(a, c_1) + \mu(a, c_2)] = 1.$

And there is the answer.