1. A Solution to Diestel, Exercise 1.8

1.1. The problem.

Find a good lower bound for the order of a connected graph in terms of its diameter and minimum degree.

1.2. My solution.

Let D be the diameter and δ the minimum degree, and let $n_0(D,\delta)$ be the exact minimum order of a graph G that meets the requirements of the problem. I'll assume $D \geq 2$ because $D = 1$ is a special case that seems irregular; it implies $G = K_2$ and $\delta = 1$. Thus, $\delta \geq 2$ also.

First, I'll get a lower bound on n_0. G must contain a pair of vertices v_0 and v_D of distance $d(v_0, v_D) = D$ and a connecting path $P = v_0 v_1 \cdots v_D$ of length D. Then P has order $D+1$, Let $O := V(G \setminus P)$ be the set of vertices in G that are not on P.

Each vertex in P has degree $d(v_i) \geq \delta$, so the number of neighbors in O is at least $\delta - 1$ for v_0 and v_D and $\delta - 2$ for v_i with $0 < i < D$. That gives a total of at least $2(\delta-1)+(D-1)(\delta-2)$ for the number of edges from vertices in P to vertices in O. But how many times can a vertex in O be counted this way? If $v \in O$, then v can at most have three neighbors in P, namely v_{i-1}, v_i, v_{i+1} for some i, because otherwise v will give a shorter path between v_0 and v_D, which is impossible by assumption. Therefore, the number of edges between P and O is at most $3|O|$. Therefore, $2(\delta-1) + (D-1)(\delta-2) \leq 3|O|$, or

$$|O| \geq \frac{1}{3}\big((D+1)(\delta-2)+2\big).$$

That implies

$$|G| = D + 1 + |O| \geq \frac{1}{3}\big((D+1)(\delta+1)+2\big).$$

That tells us that

$$n_0(D,\delta) \geq \left\lceil \frac{(D+1)(\delta+1)+2}{3} \right\rceil. \tag{1.1}$$

Second, we need an example of a graph G to see how small n_0 might be. Let's join a $K^{\delta-1}$ to v_0, v_1, v_2.

If $D = 2$ that gives minimum degree δ with $|G| = \delta + 2$. (By the way, this graph is $G = K^{\delta-1} \setminus e$.) Thus, $n_0(2,\delta) \leq \delta + 2$. This is the same as the lower bound in Equation (1.1) so the exact answer in this case is

$$n_0(2,\delta) = \delta + 2. \tag{1.2}$$

If $D > 2$ we need more neighbors of v_D, so join a $K^{\delta-1}$ to v_D, v_{D-1}, v_{D-2}. If $3 \leq D \leq 5$ that gives minimum degree δ with $|G| = D + 1 + 2(\delta-1)$.

If $D > 5$ we have middle vertices $v_3, \ldots, v_{D-3} \in V(P)$ that need $\delta - 2$ more neighbors. There are $\lceil \frac{1}{3}(D-5) \rceil$ groups of 3 of these vertices so we need that many $K^{\delta-2}$ graphs joined to those groups of 3, which gives us $\lceil \frac{1}{3}(D-5) \rceil (\delta-2)$ additional vertices in G. That makes a total of

$$D + 1 + 2(\delta-1) + \left\lceil \frac{D-5}{3} \right\rceil (\delta-2) = \left\lceil \frac{D+1}{3} \right\rceil (\delta+1) + 2$$

vertices in G. (We don't need any more vertices, because all the vertices in the added cliques have at least 3 neighbors in P, giving them degree not less than $(\delta-2)+3 > \delta$, at worst.)

Therefore,

$$n_0(D,\delta) \le \left\lceil \frac{D+1}{3} \right\rceil (\delta+1) + 2 \tag{1.3}$$

when $D \ge 6$. This same formula applies to the cases where $D = 3, 4, 5$ (since then $\lceil \frac{1}{3}(D+1) \rceil = 2$). Thus, the bounds are

$$\left\lceil \frac{(D+1)(\delta+1)+2}{3} \right\rceil \le n_0(D,\delta) \le \left\lceil \frac{D+1}{3} \right\rceil (\delta+1) + 2 \tag{1.4}$$

for $D \ge 3$.

1.3. **How good?**

How close are the upper and lower bounds? Well,

$$\left\lceil \frac{(D+1)(\delta+1)+2}{3} \right\rceil \ge \frac{(D+1)(\delta+1)+2}{3}$$

and

$$\left\lceil \frac{D+1}{3} \right\rceil (\delta+1) + 2 \le \frac{D+3}{3}(\delta+1) + 2,$$

so the difference between the lower and upper bounds is

$$\le \left(\frac{D+3}{3}(\delta+1) + 2 \right) - \left(\frac{(D+1)(\delta+1)+2}{3} \right) = \frac{2}{3}(\delta+3).$$

That means my upper and lower bounds differ by not more than approximately $2\delta/3$, no matter how large the diameter gets. This is not bad.

It's possible that the difference between my lower and upper bounds is not that big; I did not analyze the effect of the ceiling functions carefully.

It's also possible that the bounds can be improved, but this is certainly a good enough answer to the problem.