Short Summary of Basics of Equiangular Lines

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The goal is to find the most possible equiangular lines in a given dimension d. We're not too concerned about the angle itself.

There are two parts to this summary. First, we go from equiangular lines to the Seidel matrix of a graph. Then we reverse the process, showing how to begin with a graph and find out what it tells us about equiangular lines.

1. From equiangular lines to graphs

1. Suppose we have a family $\mathcal{L} = \{l_1, \ldots, l_n\}$ of n equiangular lines in $\mathbb{R}^d$, all forming the same angle ψ with each other. Assume $d > 1$ and assume the lines span $\mathbb{R}^d$ (if not, just take the subspace they span). The angle $\psi \in (0, \pi/2]$. We assume $\psi < \pi/2$, since the case $\psi = \pi/2$ means we just have d mutually orthogonal lines in $\mathbb{R}^d$, which we know all about and is not large enough to be interesting to us. Since it's trivial to get $n = d$ equiangular lines, *we'll assume* $n > d$ (that makes the family $\mathcal{L}$ interesting).
2. Choose a unit vector x_i in each line l_i.
3. Form the Gram matrix G of inner products of the unit vectors. That is, let's say
$$M := \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix};$$
then
$$G = M^T M = \begin{pmatrix} x_1 \cdot x_1 & x_1 \cdot x_2 & \cdots & x_1 \cdot x_n \\ x_2 \cdot x_1 & x_2 \cdot x_2 & \cdots & x_2 \cdot x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n \cdot x_1 & x_n \cdot x_2 & \cdots & x_n \cdot x_n \end{pmatrix} = \begin{pmatrix} 1 & \pm\cos\psi & \cdots & \pm\cos\psi \\ \pm\cos\psi & 1 & \cdots & \pm\cos\psi \\ \vdots & \vdots & \ddots & \vdots \\ \pm\cos\psi & \pm\cos\psi & \cdots & 1 \end{pmatrix}.$$
Since the lines span $\mathbb{R}^d$, we have $\operatorname{rk} G = d$, so the nullity is $\operatorname{nul} G = n - d$. Rephrased,
$$d = n - \operatorname{nul} G.$$
Since we assumed $n > d$, 0 has positive multiplicity $n - d$.
4. The matrix $\frac{1}{\cos\psi}(G - I)$ has 0 diagonal and ± 1 off the diagonal and is symmetric, so it is the Seidel matrix of a graph X of order n; i.e.,
$$\frac{1}{\cos\psi}\big(G - I\big) = S(X).$$
5. Now we look at eigenvalues. G is positive semidefinite, since it is a Gram matrix. Therefore, its least eigenvalue $\lambda_1(G) \geq 0$. Since we assumed $n > d$, we have $\lambda_1(G) = 0$ with multiplicity $n - d$.
6. All the eigenvectors of G are also eigenvectors of I (naturally), so we can simplify the expression for λ_1, namely:
$$\lambda_1\big(S(X)\big) = \lambda_1\Big(\frac{1}{\cos\psi}\big(G - I\big)\Big) = \frac{1}{\cos\psi}\lambda_1\big(G - I\big) = \frac{1}{\cos\psi}\big(\lambda_1(G) - 1\big) = -\frac{1}{\cos\psi}$$
with multiplicity $n - d$.

7. Since $\operatorname{tr}(S(X)) = 0$ (and $S(X) \neq O$), $\lambda_1(S(X)) < 0$. So we get this conclusion about the angle:
$$\cos\psi = -\frac{1}{\lambda_1(S(X))}.$$

2. From graphs to equiangular lines

Now we reverse the process. This is where it gets interesting!

1. We start with any graph X of order n and form its Seidel matrix $S(X)$.
2. We get the least eigenvalue, $\lambda_1(S(X))$. (It's a negative number.)
3. Choose
$$\alpha = -\frac{1}{\lambda_1(S(X))}.$$
(This is a positive number because $\lambda_1 < 0$.)
4. Form
$$G := I + \alpha S(X).$$
Then $\lambda_1(G) = 1 + \alpha\lambda_1(S(X)) = 0$. Therefore, G is positive semidefinite, which means it is a Gram matrix (of unit vectors because the diagonal elements equal 1). Let the vectors be $x_1, \ldots, x_n \in \mathbb{R}^d$ for some dimension d. Thus (if $i \neq j$), $\alpha(S(X))_{ij} = \pm\alpha = x_i \cdot x_j = \cos(\psi_{ij})$, where ψ_{ij} is the angle between x_i and x_j.
   Since $\cos\psi_{ij} = \pm\alpha$, each $\psi_{ij} = \psi$ or $\pi - \psi$ for angle $\psi \in (0, \pi/2]$. (But $\psi = \pi/2$ can be ruled out because $\alpha > 0$.)
5. Since we can assume d is the dimension of the span of the vectors x_i, it is also the rank of the matrix
$$M := \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}.$$
   (Notice how we're getting the same kinds of objects as when we began with $\mathcal{L}$, but we're building them up in the opposite direction, now starting from X.)
6. Since G is the Gram matrix $M^T M$, $\operatorname{rk} G = \operatorname{rk} M = d$.
7. Let $m_1 :=$ the multiplicity of λ_1 as an eigenvalue of $S(X)$. Then m_1 is the multiplicity of 0 as an eigenvalue of G. That equals the nullity of G,[1] which equals $n - \operatorname{rk} G$. So, $m_1 = n - d$.
8. Let
$$\mathcal{L} := \{\langle x_i \rangle\}_{i=1}^n.$$
Thus, $\mathcal{L}$ is a system of n lines in $\mathbb{R}^{n-m_1}$, all of which make the same angle $\psi \in (0, \pi/2)$ with each other. We now have a system of n equiangular lines in a space of dimension $n - m_1$, which is small if the least eigenvalue of our original Seidel matrix $S(X)$ has high multiplicity.

[1] The nullity is the dimension of the null space, which is (by the definitions) the dimension of the eigenspace of 0. This is called the geometric multiplicity of 0. The algebraic multiplicity is its multiplicity as a zero of the characteristic polynomial. Theorem: They are equal for symmetric matrices. That is why $m_1 = \dim(\text{eigenspace})$.

3. Example

Let's look at the example of $\overline{L(K_8)}$ in the book, and also $\overline{L(K_n)}$ in general. (Note that this n is different from the previous n, which is now $\binom{n}{2}$ since we're in the complemented line graph.) $\overline{L(K_8)}$ gives us $21 = \binom{7+1}{2}$ lines in $\mathbb{R}^7$, which is the most possible by the absolute bound. Why is that? Good luck, that's why!

The eigenvalues of $S(\overline{L(K_n)})$ and their multiplicities are

Eigenvalue	Multiplicity	Eigenvector(s)
$-\frac{1}{2}(n-2)(n-7)$	1	$\mathbf{1}$
$2n-7$	$n-1$	$x \perp \mathbf{1}$
-3	$\binom{n-1}{2}-1$	$x \perp \mathbf{1}$

(This assumes $n \geq 3$ to avoid trivial cases.) The least eigenvalue λ_1, with multiplicity m_1, is

$$(\lambda_1, m_1) = \begin{cases} (-3, \binom{n-1}{2}-1) & \text{if } n \leq 8, \\ (-\frac{1}{2}(n-2)(n-7), 1) & \text{if } n \geq 8. \end{cases}$$

For $n > 8$, $m_1 = 1$ and $\overline{L(K_n)}$ gives us $\binom{n}{2}$ equiangular lines in $\mathbb{R}^{\binom{n-1}{2}-1}$, which is unimpressive. For $3 < n < 8$, $m_1 = \binom{n-1}{2} - 1$ so we get $\binom{n}{2}$ equiangular lines in $\mathbb{R}^n$; this is the absolute bound for dimension $n-1$ but we're in dimension n, so we don't achieve the absolute bound for our dimension.

However, for $n = 8$ the first and last eigenvalues are both equal to the same number -3, whose multiplicity is $\binom{n-1}{2} - 1$, so the multiplicities add and we get $m_1 = \binom{n-1}{2} = 21$ and dimension $d = n - 1 = 7$. Thus, we achieve the absolute bound by sheer good luck.

Eigenvalues.

The eigenvalues of $A(K_n)$ and their multiplicities are

Eigenvalue	Multiplicity	Eigenvector(s)
$n-1$	1	$\mathbf{1}$
$2n-7$	$n-1$	$x \perp \mathbf{1}$

The eigenvalues and multiplicities of $A(L(K_n))$ are then

Eigenvalue	Multiplicity	Eigenvector(s)
$2(n-2)$	1	$\mathbf{1}$
$n-4$	$n-1$	$x \perp \mathbf{1}$
-2	$\binom{n-1}{2}-1$	$x \perp \mathbf{1}$

From this I get the eigenvalues of the Seidel matrix $S(L(K_n))$. Since $S(\overline{L(K_n)}) = -S(\overline{L(K_n)})$, I negate those eigenvalues.