Equiangular Lines from Complete-Graph Line Graphs
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Trying to figure out the example of 20 (= many) equiangular lines in $\mathbb{R}^7$ (from $\overline{L(K_8)}$, Godsil & Royle, pages 250–251) led me to examine how to get equiangular line sets from $L(K_n)$ and its complement $\overline{L(K_n)}$.

This write-up is supposed to be a detailed explanation of all the steps, which were not all given in detail by Godsil & Royle.

1. Equiangular lines from strongly regular graphs

There are two steps in going from a strongly regular graph $X = \mathrm{SRG}(v,k,a,c)$ to a set of v equiangular lines.

Step 1: Start with a strongly regular graph X.

Step 2: Convert $A(X)$ to the Seidel matrix $S(X)$, i.e., the adjacency matrix of (K_n, σ) where $\Sigma^- = X$.

Step 3: Use $S(X)$ to get the angle and dimension of the set $\mathcal{L}$ of equiangular lines.

Step 4: Get the unit vectors that generate the lines. (We don't pay too much attention to this step. It's matrix theory.)

1.1. **Start with X.**

Assume X and $\overline{X}$ are connected. We know the eigenvalues and multiplicities:

k with multiplicity 1,

$\theta = \frac{1}{2}[a - c + \sqrt{\Delta}]$ with multiplicity $m_\theta = \frac{1}{2}\big[v - 1 - \dfrac{2k + (v-1)(a-c)}{\sqrt{\Delta}}\big]$,

$\tau = \frac{1}{2}[a - c - \sqrt{\Delta}]$ with multiplicity $m_\tau = \frac{1}{2}\big[v - 1 + \dfrac{2k + (v-1)(a-c)}{\sqrt{\Delta}}\big]$,

where $\Delta = (a-c)^2 + 4(k-c)$.

1.2. **Convert the matrix.**

Write $A := A(X)$, $\overline{A} := A(\overline{X})$, $S := S(X)$, and $S(\overline{X}) = -S$. Since $S = \overline{A} - A$, we get $S = J - I - 2A$.

Consider the eigenvectors:

$\mathbf{1}$ gives $S\mathbf{1} = J\mathbf{1} - \mathbf{1} - 2A\mathbf{1} = (v - 1 - 2k)\mathbf{1}$.

$x \perp \mathbf{1}$ gives $Sx = Jx - x - 2Ax = (0 - 1 - 2\theta)x$ or $(0 - 1 - 2\tau)x$, depending on which eigenvalue is associated to x.

So, the eigenvalues follow the pattern:

$k \mapsto k' = v - 1 - 2k$,
$\theta \mapsto \theta' = -1 - [a - c + \sqrt{\Delta}]$,
$\tau \mapsto \tau' = -1 - [a - c - \sqrt{\Delta}]$,

with the same multiplicities.

There is a possible trick here. Note that $\tau' > \theta'$ (if $\Delta \neq 0$) so the smallest eigenvalue is either k' or θ', but k' might turn out to equal θ', thereby increasing the multiplicity of the least eigenvalue to $m_\theta + 1$.

1.3. **Get the parameters of $\mathcal{L}$.**

That means find the least eigenvalue of S, say $\lambda_1(S)$ (necessarily negative), and its multiplicity m_1. Then form the matrix $S - \lambda_1 I$, then normalize the diagonal to I by going to $I - \frac{1}{\lambda_1}S$. This is a positive semi-definite matrix so it is a Gram matrix of vectors, i.e., the columns of a matrix M such that $M^T M = I - \frac{1}{\lambda_1}S$. The dimension is the rank of M, which equals the rank of $M^T M$; therefore, the latter is $v - \mathrm{nul}(M) = v - m_1$.

The value of λ_1 is k' or θ' and the multiplicity is 1, m_θ, or $m_\theta + 1$, depending on how things work out.

Conclusion: We get v lines in $\mathbb{R}^{v-m_1}$ with angle $\arccos(-\frac{1}{\lambda_1})$.

1.4. **Get the parameters of $\overline{\mathcal{L}}$.**

We could have used $-S = S(\overline{X})$ instead to produce a different set $\overline{\mathcal{L}}$ of equiangular lines. (This simple relationship between $S(\overline{X})$ and $S(X)$ is a great help in examples.) The eigenvalues are $-k'$, $-\theta'$, $-\tau'$ with the same multiplicities as before. We know $-\theta' > -\tau'$ (if $\Delta \neq 0$)) so the least eigenvalue is either $-k'$ or $-\tau'$, but as before it's possible that $-k' = -\tau'$, so the least eigenvalue is either $-k'$ or $-\tau'$ and its multiplicity might turn out to be 1, m_τ, or $m_\tau + 1$.

2. Apply to the line graph of the complete graph

Now we look at the example $X = L(K_n) = \mathrm{SRG}(\binom{n}{2}, 2(n-2), n-2, 4)$ and the equiangular line families $\mathcal{L}_n$ that can be obtained from $L(K_n)$ and $\overline{\mathcal{L}}_n$ from $\overline{L(K_n)}$. This will include both $L(K_8)$ with its line family $\mathcal{L}_8$ and also the original example, $\overline{L(K_8)}$, with its line family $\overline{\mathcal{L}}_8$.

Here $\Delta = (n-6)^2 + 4([2n-4]-4) = (n-2)^2$, which makes $\sqrt{\Delta} = n-2$, a nice number and definitely nonzero. Let's assume $n \geq 3$, since when $n \leq 2$ the line graph is totally trivial. Therefore, $\theta > \tau$ so $\theta' < \tau'$.

The eigenvalues of $L(K_n)$ are

$k = 2(n-2)$ with multiplicity 1,
$\theta = \frac{1}{2}[n - 6 + (n-2)] = n - 4$ with multiplicity

$$m_\theta = \frac{1}{2}\Big[\binom{n}{2} - 1 - \frac{4(n-2) + (\binom{n}{2} - 1)(n-6)}{\sqrt{\Delta}}\Big] = n - 1,$$

$\tau = \frac{1}{2}[n - 6 - (n-2)] = -2$ with multiplicity

$$m_\tau = \frac{1}{2}\Big[\binom{n}{2} - 1 + \frac{4(n-2) + (\binom{n}{2} - 1)(n-6)}{\sqrt{\Delta}}\Big] = \binom{n-1}{2} - 1.$$

2.1. **Get the parameters of $\mathcal{L}_n$.**

The eigenvalues of $S = S(L(K_n))$ are

$k' = \binom{n}{2} - 1 - 4(n-2) = \binom{n}{2} - 4n + 7$,
$\theta' = -1 - 2(n-4) = -2n + 7$,
$\tau' = -1 - 2(-2) = 3$,

with the same multiplicities, though now some eigenvalues might be equal.

We have to decide what $\lambda_1(S)$ is and its multiplicity. Since $\theta' < \tau'$, $\lambda_1(S)$ is the smaller of k' and θ'. A calculation shows that

$$\begin{cases} k' < \theta' & \text{when } n < 5, \\ k' = \theta' & \text{when } n = 5, \\ k' > \theta' & \text{when } n > 5. \end{cases}$$

Therefore,

$$\lambda_1(S) = \begin{cases} k' = \binom{n}{2} - 4n + 7 & \text{when } n < 5, \text{ with multiplicity } m_1 = 1, \\ k' = \theta' = -3 & \text{when } n = 5, \text{ with multiplicity } m_1 = \binom{n-1}{2} = 6, \\ \theta' = -2n + 7 & \text{when } n > 5, \text{ with multiplicity } m_1 = \binom{n-1}{2} - 1. \end{cases}$$

This gives us $|\mathcal{L}_n| = \binom{n}{2}$ lines in dimension $v - m_1 = n$ when $n > 5$, 10 lines in dimension $v - m_1 = 4$ when $n = 5$, and hardly any lines in dimension $\binom{n}{2}$ when $n = 3, 4$. The number of lines in dimension $n > 5$ is close but not quite enough to meet the absolute bound, so we can't decide this way whether the absolute bound is attainable for $n > 5$.

In dimension $d = 5$, though, we have exactly $\binom{d+1}{2} = 10$ lines, proving that the absolute bound can be attained in that dimension. The angle of these lines is $\arccos \frac{1}{3}$.

2.2. **Get the parameters of $\overline{\mathcal{L}}_n$.**

The eigenvalues of $S(\overline{L(K_n)}) = -S$ are

$\bar{k}' = -k' = -\binom{n}{2} + 4n - 7,$
$\bar{\theta}' = -\theta' = 2n - 7,$
$\bar{\tau}' = -\tau' = -3,$

with the same multiplicities as with S (of course). Again, some eigenvalues might be equal; we need $\lambda_1(-S)$ and its multiplicity. Since $\bar{\theta}' > \bar{\tau}'$, the least eigenvalue is the smaller of $\bar{k}'$ and $\bar{\tau}' = -3$; a calculation gives

$$\begin{cases} \bar{k}' > \bar{\tau}' & \text{when } n < 8, \\ \bar{k}' = \bar{\tau}' & \text{when } n = 8, \\ \bar{k}' < \bar{\tau}' & \text{when } n > 8. \end{cases}$$

Therefore,

$$\lambda_1(S) = \begin{cases} \bar{\tau}' = -3 & \text{when } n < 8, \text{ with multiplicity } m_1 = \binom{n-1}{2} - 1, \\ \bar{k}' = \bar{\tau}' = -3 & \text{when } n = 8, \text{ with multiplicity } m_1 = \binom{n-1}{2} = 21, \\ \bar{k}' = -\binom{n}{2} + 4n - 7 & \text{when } n > 8, \text{ with multiplicity } m_1 = 1. \end{cases}$$

The number of lines we get this way is $|\overline{\mathcal{L}}_n| = \binom{n}{2}$ lines in dimension n when $n = 3, 4, \ldots, 7$, 28 lines in dimension $28 - 21 = 7$ when $n = 8$ (this is the case in the book), and $\binom{n}{2}$ lines in dimension $\binom{n}{2} - 1$ for $n > 8$ (a pathetically small number for that dimension, since the coordinate axes already give us $\binom{n}{2} - 1$ lines in that dimension).

You can see the dramatic effect of the coincidence of equality of eigenvalues of $S(\overline{L(K_8)})$ (and for $S(L(K_5))$). We just accidentally, as it were, achieve the absolute bound. The lines in $\overline{\mathcal{L}}_8$ have angle $\arccos \frac{1}{3}$; curiously, the same as in $\mathcal{L}_5$.

For $3 \leq n \leq 7$ we don't get that close, because the dimension doesn't happen to be quite low enough. For $n > 8$ we get nothing of interest, at least for getting the most lines.