The Shrikhande Graph

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Questions in mathematical statistics led to the discovery of some elegant mathematical structures and the Shrikhande graph is a beautiful example of the case in point. This article is an attempt to discuss the Shrikhande graph along with its connections in algebra, group theory and topology.

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1. Introduction

S S Shrikhande discovered these graphs while investigating certain uniqueness questions related to Latin squares and association schemes. He showed that uniqueness was valid apart from an exception which gave rise to what are now called Shrikhande graphs. Historically, the configurations similar to those discussed in this exposition followed a fruitful interaction between the requirements of the statistical field of design [1] of experiments on the one hand and the contributions to it from finite fields and geometries on the other [2]. This is illustrated in the following anecdotal quotation on R C Bose [3]:

This month marks the 98th birthday of S S Shrikhande.

"There is a famous joke about Bose's work in Giridih. Professor Mahalanobis wanted Bose to visit the paddy fields and advise him on sampling problems for the estimation of yield of paddy. Bose did not very much like the idea, and he used to spend most of the time at home working on combinatorial problems using Galois fields. The workers of the ISI used to make a joke about this. Whenever Professor Mahalanobis asked about Bose, his secretary would say that Bose is working in fields, which kept the Professor happy."

Keywords
Shrikhande graph, design of experiments, permutation groups, combinatorial topology.

This expository article is a leisurely attempt to discuss

If Γ is a graph and Γ' its subgraph, then Γ' is called a clique if it is a complete graph.

the Shrikhande graph in the modern-day framework and terminology along with its connections in algebra, group theory and topology [4].

2. Preliminaries

A graph Γ is a pair (V, E) where V is a (finite) set of vertices and and E (called the edge set) consists of some unordered pairs (xy) of vertices with $x \neq y$. We write $e = (xy)$ to denote the edge e that has x and y as the end-vertices. In this case, x (also y) is said to be incident with the edge e and the vertices x and y are said to be adjacent to each other if (xy) is an edge. We frequently write $x \sim y$ when there is an edge $e = (xy)$ (between x and y). A drawing of Γ is a picture of Γ in the plane. Degree of a vertex is the number of edges incident at that vertex and Γ is called a k-regular graph if every vertex has degree k. A subgraph Γ' of Γ has some of the vertices and edges of Γ with (xy) allowed to be an edge in Γ' only when it is also an edge in Γ. Further, Γ' is called an induced subgraph if x, y are vertices in Γ' such that x and y are adjacent in Γ, then they must also be adjacent in Γ'. A complete graph K_n has n vertices with every two vertices adjacent (thus it is a regular graph of degree $n-1$). If Γ is a graph and Γ' its subgraph, then Γ' is called a clique if it is a complete graph. For example, every induced subgraph of a complete graph is a clique. The drawings of some graphs are shown in *Figure* 1. The first is called C_4 (the cycle on 4 vertices), the second the kite and the other two are two different drawings of the complete (all possible edges) graph K_4.

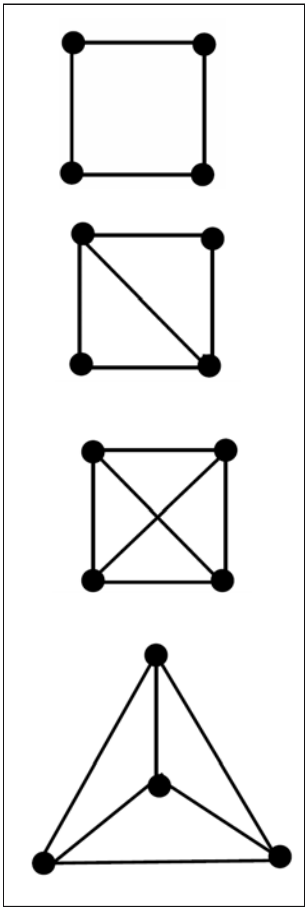

Figure 1. C_4, kite and two drawings of K_4.

An automorphism α of a graph Γ is a permutation of its vertex set which preserves adjacencies [5]. Thus, for the complete graph K_4 any permutation will qualify to be an automorphism while this is not the case for the first two graphs. As functions (on the vertex set) we may compose two automorphisms and hence we can talk of an automorphism group to be a group of (some) automorphisms of Γ. The full automorphism group of Γ is the set of all automorphisms of Γ and we call it the *automorphism group* of Γ. Thus, S_4 is the automorphism group of K_4. The automorphism group of C_4 is the dihedral group D_4 of order 8 that consists of (treating the graph as a square) 4 rotations and 4 reflections (this is also true for any cycle C_n with $n \geq 3$ where the automorphism group is D_n). The automorphism group of the kite graph is, in fact, smaller and is isomorphic to the Klein group V_4 which is of order four and consists of two reflections and a rotation through 180° besides identity (we cannot map a vertex of degree 2 to a vertex of degree 3). Isomorphic graphs clearly have isomorphic automorphism groups.

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In a general setup, we have a group G that acts as a permutation group on a set X and we write xRy if we have an $\alpha \in G$ such that $\alpha(x) = y$. This is an equivalence relation and the equivalence classes of X under the G-action are called its *orbits*. The orbit of x is denoted by x^G and it is the equivalence class of x under the G-action. For $x \in X$, the set $G_x = \{\alpha \in G : \alpha(x) = x\}$ is a subgroup of G, called the stabilizer of x in G. A basic relationship between the orbit size and the stabilizer is given by the following:

Orbit Stabilizer Lemma

$$|G_x| \times |x^G| = |G|.$$

G is said to be transitive[1] *(on X) if it has only one orbit.*

For example, the automorphism group of C_4 (which is D_4) is transitive both on the vertex and edge sets of the graph. For the kite graph, we have two orbits on the vertex set and also two orbits on the edge set. Let G be a group that acts *transitively* on X. It is possible to

[1] G is said to be transitive on X if every point in X can be transformed into another point of X by an element of G.

If the number of orbits of G on $X \times X$ is r, then G is said to have rank r action (on X).

look at the G-action on the Cartesian product $X \times X$: If $(x, y) \in X \times X$ and if $\alpha \in G$, then $\alpha(x, y) = (\alpha(x), \alpha(y))$. This is also a permutation action and hence has orbits. Since G is assumed to be transitive, one orbit is the diagonal $D = \{(x, x) : x \in X\}$. If the number of orbits of G on $X \times X$ is r, then G is said to have rank r action (on X). Now consider the subgroup $H = G_x$. Then the action of H on X, has the singleton $\{x\}$ as an orbit and we have a bijective correspondence between orbits of H (on X) and those of G (on $X \times X$) which allows one to define the rank in an alternative manner as the number of H-orbits on X. A (transitive) group with rank 2 is precisely a 2-transitive group (any ordered pair can be mapped to any ordered pair). An orbit O is called symmetric if $(x, y) \in O$ implies $(y, x) \in O$ (it is enough to check this for only one pair in O). Non-diagonal orbits are paired in a natural way (by transpose). In particular, if G has rank 3 with symmetric orbits then it allows us to construct nice graphs. Before closing this discussion, we note a fact which can be proved easily: if G is rank 3 and has an element of order two (which is equivalent to having $|G|$ even), then both the non-diagonal orbits are symmetric.

3. Strongly Regular Graph

The notion of a strongly regular graph Γ defined by R C Bose [6] was a culmination of some earlier work including that of S S Shrikhande which is the main theme of this exposition. It is a more special case of the concept of an association scheme (also defined by Bose earlier) and its usefulness and centrality can be judged from the fact that it conveniently captures requirements of several disciplines including algebra, group theory and coding theory. A simple k-regular graph Γ on n vertices is called a *strongly regular graph* (or SRG for short), if there are constants a and c such that a pair (x, y) of vertices is commonly adjacent to a (respectively to c) vertices if x and y themselves are adjacent (respectively

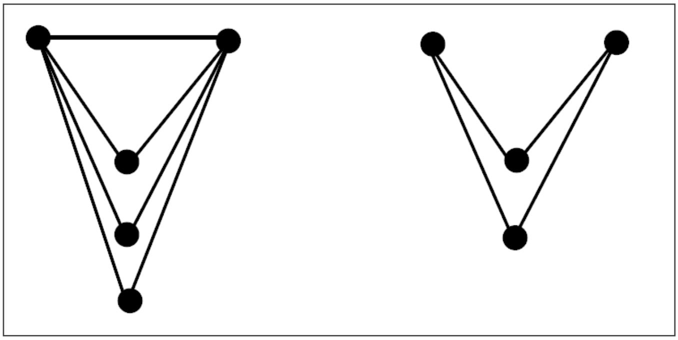

Figure 2. Parameters a = 3 and c = 2 of an SRG.

not adjacent) (*Figure* 2).

Call such a strongly regular graph an (n, k, a, c)-SRG. As a small example, the cycle C_5 is a an $(5, 2, 0, 1)$-SRG. To avoid trivialities we also generally insist that both Γ and its complement (obtained by interchanging edges and non-edges) be connected. One of the best known examples of an SRG is the Petersen graph $\mathbf{P}$, drawn in *Figure* 3 and I cannot resist the temptation of describing it. Since this graph is a quotient[2] of the graph of a regular dodecahedron (with the identification of opposite vertices), it has a drawing on the real projective plane without edges crossing each other. Yet another way of describing it is through the symmetric group S_5 that has 10 Sylow 3-subgroups[3] taken as vertices of $\mathbf{P}$ with two vertices joined if the fixed points of the corresponding Sylow subgroups are disjoint sets. This makes

[2] A quotient of a graph G is a graph whose vertices are blocks of a partition of the vertices of G and two blocks are adjacent in the quotient graph if there are vertices in these blocks which are adjacent in the original graph.

[3] For a prime p, a p-subgroup of a finite group is a subgroup with size a power of p; it is called a Sylow p-subgroup if its size is the power of p dividing the order of G.

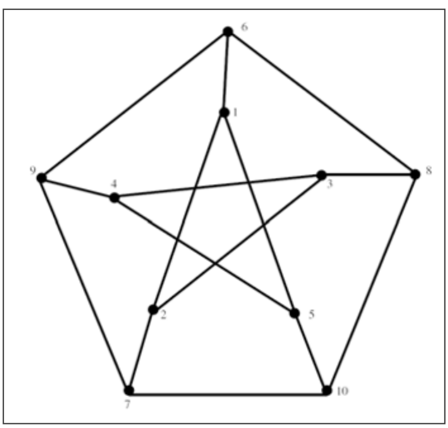

Figure 3. The Petersen graph.

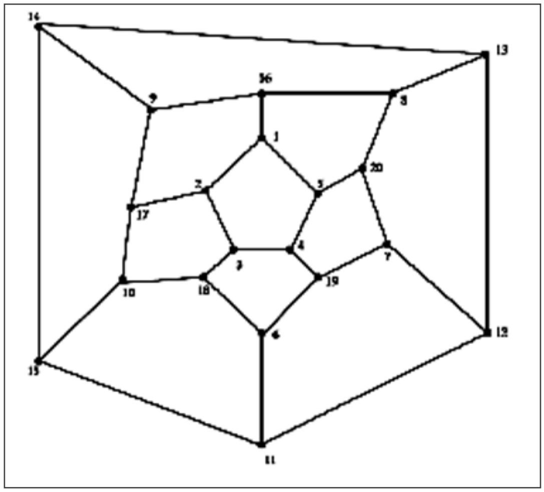

Figure 4. Skeleton of a regular dodecahedron with antipodal vertices.

it possible to view the full automorphism group of $\mathbf{P}$, which is S_5 itself acting as a rank 3 group on $\mathbf{P}$, which is a $(10, 3, 0, 1)$-SRG.

Figure 4 is a planar drawing of the skeleton of a regular dodecahedron with antipodal point-pair (points which are at opposite ends of a diameter of the encompassing circle) given by $(i, 10 + i)$. Note that the antipodal pair $(i, 10 + i)$ in *Figure* 4 is identified with a single vertex i in *Figure* 3 (where i takes values from 1 to 10).

A small and equally important graph is the Clebsch graph $\mathbf{C}$ with parameters $(16, 5, 0, 2)$. Many constructions of $\mathbf{C}$ are available though the one we give here is purely combinatorial. Take a set X of order 5 and take as the vertex set all the subsets A whose orders are even (these are 16 in number). Make two vertices A and B adjacent if the symmetric difference $A \triangle B$ is a set of order 4 (recall that $A \triangle B$ denotes the set of all those elements that are either in A or B but not both). The Clebsch graph $\mathbf{C}$ can also be obtained as a quotient of a 5-dimensional hypercube with identification of antipodal vertices as in the case of the Petersen graph. In passing we note that if x is a vertex of $\mathbf{C}$ then the induced subgraph on the set of 10 vertices not adjacent to x is isomorphic to the Petersen graph.

If x is a vertex of the Clebsch graph, then the subgraph on the set of 10 vertices not adjacent to x is isomorphic to the Petersen graph.

A rank 3 permutation group G described in the last section with symmetric non-diagonal orbits gives rise to an SRG Γ on which G acts as an automorphism group that is transitive both on the vertex and edge set of Γ. The topic of SRGs is too vast to be discussed in this exposition. For a graph Γ with n vertices, label the vertices of Γ by numbers from 1 through n. The adjacency matrix $A = [a_{i,j}]$ of Γ is a $(0, 1)$ matrix with $a_{i,j} = 1$ if and only if $i \sim j$. Then A is a real symmetric matrix with zero diagonal. The (real) vector space spanned by powers of A is called the adjacency algebra of the graph Γ and various graph properties can be translated into this algebra.

The friendship theorem asserts that if friendship in a community is a symmetric relation, and if every pair in the community has exactly one common friend, then there is a person in the community who is a friend of all the rest.

Among some nice applications of this theory is the following result called the friendship theorem: Assume that friendship in a community is a symmetric relationship. If any two people in the community have exactly one common friend, then there is one person (leader) who is a friend of all the rest with the resulting graph exactly as shown in *Figure* 5. This is called a windmill graph. We also note before closing this general discussion that among the family of regular and connected graphs, the graphs in the family of SRGs are characterized by having exactly three distinct eigenvalues of the adjacency matrix.

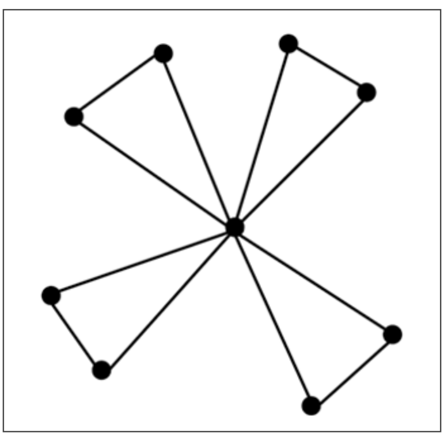

Figure 5. A windmill graph.

Bose and Shrikhande called these graphs the L_2 graphs.

Here is the family of SRGs pertinent to our discussion of the Shrikhande graph. Bose and Shrikhande called these graphs the L_2 graphs. Let $m \geq 2$ be a natural number and let M denote the set $\{1, 2, \ldots, m\}$. The vertex set of the graph $L_2(m)$ is $M \times M$, the Cartesian product with two vertices (i, j) and (i', j') adjacent if $i = i'$ or $j = j'$. This is exactly the graph on the cells of an $m \times m$ chess board with two cells declared adjacent if they are in the same row or in the same column. We then have $k = 2(m-1), a = m-2$ and $c = 2$ as the other parameters. Suppose we have an SRG Γ with the same parameters

$$(n, k, a, c) = (m^2,\ 2(m-1),\ m-2,\ 2)$$

as that of $L_2(m)$. Must Γ be necessarily isomorphic to $L_2(m)$? This was answered by S S Shrikhande who showed that the answer is in the affirmative except when $m = 4$; the next section gives Shrikhande's solution to this question.

4. Pseudo-L_2 Scheme

Shrikhande's Theorem. *Let Γ be an SRG with*

$$(n, k, a, c) = (m^2,\ 2(m-1),\ m-2,\ 2)\,.$$

Then $\Gamma \cong L_2(m)$ except when $m = 4$.

In the case $m = 4$, we have exactly one graph $\mathbf{S}$ with the same parameters as that of $L_2(4)$ (Shrikhande called this pseudo-L_2 scheme) but not isomorphic to $L_2(4)$.

The graph $\mathbf{S}$ arising as an exceptional case when $m = 4$ in Shrikhande's theorem is now called the *Shrikhande graph.*

Proof of Shrikhande's Theorem [7]. The case $m = 2$ is too small and it is easy to check that we must have $\Gamma = C_4$. Let $m = 3$. In this case, use the fact that a pair of adjacent vertices have exactly one common adjacency. Let x be a vertex and let $N(x)$ denote its neighbourhood.

This has 4 vertices forming a pair of disjoint edges and let these be y_1y_2 and z_1z_2 (there are no other adjacencies in $N(x)$ since $a = 1$). The remaining 4 vertices in Γ must have the property that each is adjacent to exactly one y and one z (use $a = 1$ and $c = 2$) and hence can be labeled v_{ij} where v_{ij} is adjacent to y_i and z_j. Since the local picture at any neighbourhood must consist of disjoint union of two edges, it follows that the vertices can be arranged in a 3×3 array

$$\begin{array}{ccc} x & z_1 & z_2 \\ y_1 & v_{11} & v_{12} \\ y_2 & v_{21} & v_{22} \end{array}$$

where two vertices are adjacent precisely when they are in the same row or column making the graph isomorphic to $L_2(3)$.

So the induced subgraph on N is a disjoint union of two copies of $K_{m\text{-}1}$.

Let $m \geq 4$. Let $x \in V$, the vertex set of V and let $N = N(x)$. For $y \in N$, let A denote the set $(N \cap N(y)) \cup \{y\}$ and let $A' = A - \{y\}$. Since $a = m - 2$, we have $|A'| = m - 2$ and $|A| = m - 1$. Let B denote the complement of A in N and let $z \in B$. Let $B' = B - \{z\}$. Since $c = 2$ and since y and z are both adjacent to x, we see that z is adjacent to, at the most, one vertex in A. But $|N \cap N(z)| = m - 2$, and hence of the $m - 2$ vertices in B', z must be adjacent to at least $m - 3$. So B' has at the most one vertex w not adjacent to z and by interchanging the roles of w and z, we obtain w not adjacent to at the most one vertex in B (which has to be z). We have exactly one of the following two cases.

Case 1: There are two vertices z and w in B that are not adjacent and hence are adjacent to every other vertex of B. Since $B - \{z, w\}$ has $m - 3$ vertices and since z and w are already adjacent to x, it follows that $m - 3 \leq 1$, thus forcing $m = 4$ as the only possibility. In this situation, B has 3 vertices z, u and w with u adjacency to both z and w with no adjacency between z and w.

Case 2: Every pair of vertices in B is adjacent. We

The graph Γ is thus isomorphic to $L_2(m)$.

then have a clique K_{m-1} on B. Since a vertex of B is adjacent to precisely $m-2$ vertices of N it follows that no vertex of B is adjacent to any vertex of A and turning this around the adjacencies of any vertex of A (to the vertices in N) are in A. Since $|A| = m-1$ and since $a = m-2$ we also have a clique K_{m-1} on A. So the induced subgraph on N is a disjoint union of two copies of K_{m-1}. Label the vertices of A by $\{y_1, y_2, \ldots, y_{m-1}\}$ and those of B by $\{z_1, z_2, \ldots, z_{m-1}\}$. Notice that for any vertex u of N, the induced subgraph on $N(u)$ that has $2(m-1)$ vertices has one copy of K_{m-1} and hence using the same argument made earlier with x and N replaced by u and $N(u)$ respectively shows that $N(u)$ is also a disjoint union of two copies of K_{m-1}. A vertex v which is not in $\{x\} \cup N$ is adjacent to a unique pair (y_i, z_j) and hence can be labeled v_{ij}. We can thus write all the vertices in an $m \times m$ array: rows are indexed by $x, y_1, y_2, \ldots, y_{m-1}$ (and take this as column 0) and columns are indexed by $x, z_1, z_2, \ldots, z_{m-1}$ (and take this as row 0) with v_{ij} placed at the (i,j)-th position in the array, where two vertices are adjacent exactly when they are in the same row or column. The graph Γ is thus isomorphic to $L_2(m)$.

We are thus left with an exceptional graph Γ with parameters $(n, k, a, c) = (16, 6, 2, 2)$. Consider the induced subgraph H on $N(x)$ where x is some vertex. This is a 2-regular graph on 6 vertices and hence is either a disjoint

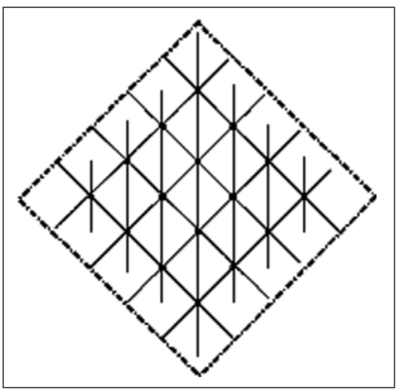

Figure 6. The Shrikhande graph drawn on a torus.

union of two K_3's or the cycle C_6. In the former case, we can use the argument made in *Case* 2 to conclude that Γ is indeed $L_2(4)$ and hence not really exceptional. Thus the only possibility we are left with is when $N(x)$ is C_6 and this must be true at every vertex x. Such a graph (the Shrikhande graph) exists and is drawn in *Figure* 6.

6. Topology and Automorphism Groups

Some discussion on the topology and automorphism groups of the graphs under consideration will be in order. For the Lattice graph $\mathbf{L} = L_2(4)$, we can permute rows and columns independently giving a direct product of S_4 with itself and, in addition, we can also interchange rows and columns (transpose of the array) giving us the full automorphism group of $\mathbf{L}$ isomorphic to the wreath product[4] of S_4 with $\mathbb{Z}_2$ (or S_2) and hence its order is $(4!)^2 \times 2$. This group has a rank 3 action (with symmetric orbits) on the vertex set of $\mathbf{L}$ and hence naturally gives rise to the strongly regular graph $\mathbf{L}$. In order to study the Shrikhande graph $\mathbf{S}$, label the vertices of $\mathbf{S}$ by ordered pairs (i, j) with both the coordinates read modulo 4. Then $(i, j) \sim (i', j')$ precisely in the following cases: (i) $i = i'$ and $j' - j \equiv \pm 1$ or (ii) $j = j'$ and $i' - i \equiv \pm 1$ or (iii) $(i', j') \equiv (i + 1, j + 1)$ or (iv) $(i', j') \equiv (i - 1, j - 1)$. The Shrikhande graph $\mathbf{S}$ can thus be drawn on a torus as shown in *Figure* 6 and we ask the reader to suitably label the vertices so as to conform with the adjacencies described in the previous statement. Here, each vertex is adjacent to the four vertices that are (immediately) to its right, left, above and below. Besides these, the two other adjacencies are to the top right and to the bottom left (north east and south west) of the given vertex. This drawing is a topological embedding on the torus and in fact it gives a triangular tessellation (tiling) of the torus with 16 vertices, 48 edges and 32 triangular regions (faces). For a connected graph Γ, embedded on a surface of genus g, we have the following Euler–Poincaré

[4] The wreath product of two groups is a construction of a new group which we do not explain precisely here. If A,B have orders m,n respectively, then the wreath product of A with B is a group of order n m^n.

Figure 6 is a topological embedding on the torus and in fact it gives a triangular tessellation (tiling) of the torus with 16 vertices, 48 edges and 32 triangular regions (faces).

The genus of a graph is the smallest genus of the surface on which the graph can be embedded.

formula:

$$n - e + f = 2 - 2g \quad , \tag{1}$$

where n is the number of vertices, e is the number of edges, f is the number of faces and g is the genus of the surface on which the graph is embedded [8]. The genus of a graph is the smallest genus of the surface on which the graph can be embedded. Neither the graph **L** nor **S** is planar since it is easily seen from (1) (with $g = 0$) that such a graph has to have a vertex of degree at the most 5, whereas both the graphs under consideration are regular of degree 6. Thus the Shrikhande graph has genus 1. In contrast, the genus of **L** does not appear to have been precisely determined though it is at least 2 as shown by the following argument. I am indebted to Arthur T White [9] for this argument. If we suppose to the contrary that **L** has a toroidal embedding, then such an embedding has 16 vertices, 48 edges and 32 triangular regions (use (1) with $g = 1$). Notice that the given graph **L** actually has 32 (combinatorial) 3-cycles (or triangles) with 4 coming from each copy of K_4 in a row or column of the 4×4 array. In particular, the four combinatorial triangles contained in a K_4 must occur as faces. The union of these four triangles is therefore the boundary of a tetrahedron, which is topologically a sphere S^2. But it is easy to show (and intuitively obvious) that the sphere S^2 cannot be embedded in the torus, a contradiction.

It is possible to give an alternative argument since it is of some pedagogical value and the interested reader may look at the combinatorial questions of rotations of graphs. The combinatorial genus (which equals topological genus) of a graph Γ is defined as follows (details in [10, 11]). At every vertex v of Γ we choose a cyclic permutation σ_v of the edges incident at v and call this a local rotation at v. A rotation of Γ is a sequence R that consists of σ_v at each vertex v. The graph **L** which is regular of degree 6 has a total of $(5!)^{16}$ rotations. Let R be a rotation of Γ. Start at any vertex u and an edge

$e_1 = (uv)$ incident at that vertex. If $e_2 = (vw)$ follows e_1 in the cyclic permutation σ_v, then we move to w and then look at which edge follows e_2 in the cyclic permutation σ_w and so on. We must eventually return back to the vertex–edge pair we started with completing a closed circuit. This procedure thus partitions the set of all directed edges of Γ into closed circuits. Let $\alpha_2(R)$ denote the number of circuits. Then the combinatorial genus is obtained from the Euler–Poincaré formula (1) when we replace f by the maximum $\alpha_2(R)$. In our case with graph $\mathbf{L}$, genus $g = 1$ requires that $\alpha_2(R)$ is 32 and this is possible if every closed circuit is a triangle. On the other hand, induced subgraph $N(u)$ at any vertex (which is a union of two K_3's) disallows this: Any cyclic permutation at u must compel us to eventually move from an edge in the row of u to an edge in the column of u. If these two edges are (uv) and (uw), then the fact that v and w are not adjacent forces a circuit with at least 4 edges containing these two edges, which is a contradiction.

Since the induced subgraph on any neighbourhood of S is a C_6, the following alternative description of S is in order.

Consider the determination of the automorphism group G of the Shrikhande graph $\mathbf{S}$. The maps $(i, j) \to (i + 1, j)$ and $(i, j) \to (i, j + 1)$ are easily seen to be automorphisms of $\mathbf{S}$ giving the transitive group $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ of order 16 as an automorphism group and hence G is clearly transitive on the vertex set of $\mathbf{S}$. Using the orbit-stabilizer theorem, therefore, it suffices to determine the order of a stabilizer of some vertex. *Since the induced subgraph on any neighbourhood of* $\mathbf{S}$ *is a* C_6*, the following alternative description of* $\mathbf{S}$ *is in order.* Let u be some vertex with $N(u) = \{0, 1, 2, 3, 4, 5\}$ where $i \sim i + 1$. Read all the numbers modulo 6 giving the 6-cycle $(0, 1, 2, 3, 4, 5)$ (in that cyclic order) as the induced subgraph $N(u)$. Since 0 and 1 are commonly adjacent to one more vertex, call this new vertex (01). We thus have 6 vertices $(01), (12), (23), (34), (45), (50)$ appropriately adjacent to adjacent vertices in $N(u)$ and these

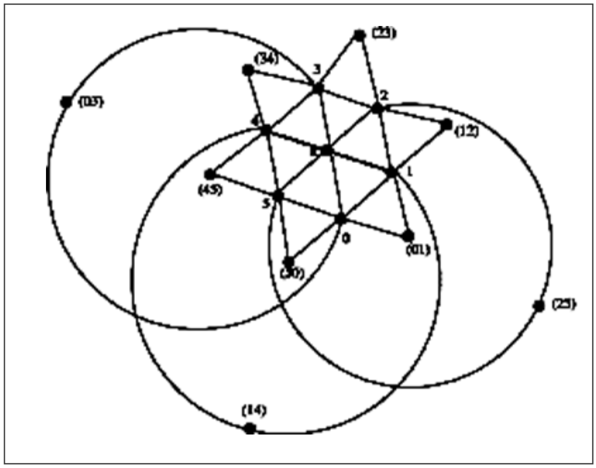

Figure 7. Local picture of the Shrikhande graph.

are all at distance 2 from u. The remaining three vertices (which are also at distance 2 from u) are $(03), (14)$ and (25) appropriately adjacent to diagonally opposite vertices in $N(u)$ (*Figure* 7).

Finally the adjacencies between the two types of vertices at distance 2 (from u) are determined uniquely using the parameters $(16, 6, 2, 2)$ along with the fact that the induced subgraph on any neighbourhood is a C_6 and these are:

$$(i(i{+}3)) \sim (i(i{+}1)), (i(i{+}5)), ((i{+}2)(i{+}3)), ((i{-}2)(i{-}3)),$$

(where $i = 0, 1, 2$). Let $H = G_u$. H must permute the vertices $0, 1, 2, 3, 4, 5$ and a very little consideration shows that $H \cong D_6$, the dihedral group on 6 letters (and of order 12) in its action on $N(u)$. It may be treated as an action on a regular hexagon. Thus $|G| = 16 \times 12 = 192$. Since the stabilizers of vertices are conjugate subgroups, the following facts about G can be verified:

(i) G has 16 Sylow 3-subgroups each with normalizer[5] isomorphic to D_6.

(ii) G has 32 elements of order three and 32 elements of order 6.

[5] The normalizer of a subgroup H in a group G consists of all g in G such that $gH = Hg$; this is the largest subgroup of G in which H is normal.

(iii) Possible orders of elements of G are $2^k, 3$ and 6 where k is some non-negative integer.

(iv) G has 3 Sylow 2-subgroups that mutually intersect in a common subgroup of order 32. In particular, G is solvable.

(v) The action of G on $\mathbf{S}$ is rank 4 with 4 symmetric orbits. These orbits are clearly visible in the second description of $\mathbf{S}$. Besides the two orbits $\{u\}$ and $N(u)$, the subgroup $H = G_u$, has two other orbits on the set of non-neigbours of $\mathbf{S}$. The Shrikhande graph $\mathbf{S}$ (unlike the graph $L_2(4)$) is distance regular[6] but is not distance transitive[7].

(vi) The maximum size of an independent set in $\mathbf{S}$ is 4 and $\{u, (03), (14), (25)\}$ is an independent set. Using the transitive action of G four such disjoint independent sets can be located in $\mathbf{S}$ to conclude that $\mathbf{S}$ has chromatic[8] number 4 (this is also true for $\mathbf{L}$).

5. Concluding Remarks

The Shrikhande graph and strongly regular graphs in general were discovered by Bose and Shrikhande purely in certain combinatorial settings where the problems involved were also of a combinatorial nature. Many interesting families of strongly regular graphs are found using elegant algebraic techniques. A combinatorial procedure called *switching* converts the lattice graph $\mathbf{L}$ into the Shrikhande graph $\mathbf{S}$ and *vice versa*. If Γ is a graph and if A is a subset of its vertex set, then switching w.r.t. A involves changing the edges into non-edges and non-edges into edges between A and its complement (the other edges and non-edges are maintained as they are). Following the description of both $\mathbf{L}$ and $\mathbf{S}$ in terms of a 4×4 array, let $D = \{(i, 4-i) : i = 0, 1, 2, 3\}$ which is an independent set for both the graphs. Switching w.r.t. D

[6] A regular graph (one in which each vertex has an equal number of neighbours) is distance regular if for any two vertices v,w the number of vertices at distance j from v and at distance k from w depends only on j,k and the distance between v,w.

[7] A distance transitive graph is one in which for any pair of points x,y at some distance i, and any other pair of vertices u,v at the same distance i from each other, there is an automorphism mapping x to u and y to v.

[8] Chromatic number of a graph is the smallest number of colours needed to colour the vertices with adjacent ones coloured differently.

converts the lattice graph **L** into the Shrikhande graph **S** and *vice versa.* Switching is a powerful technique used in graph theoretic treatment of maximum number of equiangular lines in the Euclidean space $\mathbb{R}^n$.

Suggested Reading

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