A ring in graph theory

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1. Introduction

We call a point set in a complex K a *0-cell* if it contains just one point of K, and a *1-cell* if it is an open arc. A set L of 0-cells and 1-cells of K is called a *linear graph* on K if

(i) no two members of L intersect,

(ii) the union of all the members of L is K,

(iii) each end-point of a 1-cell of L is a 0-cell of L

and (iv) the number of 0-cells and 1-cells of L is finite and not 0.

Clearly if L is a linear graph on K, then K is either a 0-complex or a 1-complex, and L contains at least one 0-cell.

A 1-cell of L is called a *loop* if its two end-points coincide and a *link* otherwise.

We say that L is connected if K is connected. If not then the subset of L consisting of the 0-cells and 1-cells of L which are in a component K_1 of K constitute a *component* of L. A component of a linear graph is itself a linear graph.

Let the numbers of 0-cells and 1-cells of a linear graph L on a complex K be $\alpha_0(L)$ and $\alpha_1(L)$ respectively. Then if $p_i(L) = p_i(K)$ is the Betti number of dimension i of K we have by elementary homology theory

$$\alpha_1(L) - \alpha_0(L) = p_1(L) - p_0(L). \tag{1}$$

Let L_1, L_2 be linear graphs on K_1, K_2 respectively. Then if there is a homoeomorphism of K_1 on to K_2 which maps each i-cell of L_1 on to an i-cell of L_2 $(i = 0, 1)$ we say that L_1 and L_2 are *isomorphic* and write

$$L_1 \cong L_2. \tag{2}$$

If L_1 and L_2 are two linear graphs whose complexes K_1 and K_2 do not meet, then together they constitute a linear graph L on the union of K_1 and K_2. We call it the product of L_1 and L_2 and write

$$L = L_1 L_2. \tag{3}$$

The set of all the 0-cells of a linear graph L, together with an arbitrary subset of the 1-cells constitutes a linear graph S which we call a *subgraph* of L. We call S a *subtree* of L if $p_0(S) = 1$ and $p_1(S) = 0$.

Let A be a link in a linear graph L on a complex K. By suppressing A we derive from L a linear graph L'_A on a complex K'_A. By identifying all the points of the closure of A in K and taking the resulting point as a 0-cell of the new linear graph we derive from L a linear graph L''_A on a complex K''_A.

Now there exist single-valued functions $W(L)$ on the set of all linear graphs to the ring I of rational integers which obey the general laws

$$W(L_1) = W(L_2) \quad \text{if} \quad L_1 \cong L_2 \tag{4}$$

and

$$W(L) = W(L'_A) + W(L''_A), \tag{5}$$

where A is any link of L. Some of these functions also satisfy

$$W(L_1 L_2) = W(L_1)\, W(L_2), \tag{6}$$

whenever the product $L_1 L_2$ exists.

We give here three examples; all three satisfy (4) and (5) and the last two satisfy (6). Proofs of these statements will emerge later, but the reader may easily verify them at once.

(I) $W(L)$ is the number of subtrees of L. This function is connected with the theory of Kirchhoff's Laws. A summary of its properties and an application of it to dissection problems is given in a paper entitled 'The dissection of rectangles into squares' by Brooks, Smith, Stone and Tutte (*Duke Math. J.* 7 (1940), 312–40). These authors call it the *complexity* of L.

(II) $(-1)^{\alpha_0(L)}\, W(L)$ vanishes whenever L contains a loop, and is otherwise equal to the number of single-valued functions on the set of 0-cells of L to some fixed set H of a finite number n of elements such that for each 1-cell of L the two end-points are associated with different elements of H.

Important papers dealing with such 'colourings of the 0-cells of L in n colours' are 'The coloring of graphs' by Hassler Whitney* (*Ann. Math.* 33 (1932), 688–718) and 'On colouring the nodes of a network' by R. L. Brooks (*Proc. Cambridge Phil. Soc.* 37 (1941), 194–97).

(III) If we orient the 1-cells of L and adopt the convention that the boundary of an oriented loop vanishes, we can define 1-cycles on L with coefficients in a fixed additive Abelian group G of finite order λ. $(-1)^{\alpha_0(L)+\alpha_1(L)}\, W(L)$ is the number of such 1-cycles on L in which no 1-cell has for coefficient the zero element of G.

These examples suggest that a general theory of functions satisfying the laws (4) and (5) should be constructed, and this paper represents an attempt to develop such a theory. For this purpose it is convenient to have the following definitions.

A *W-function* (*V-function*) is a single-valued function on the set of all linear graphs to an additive Abelian group G (commutative ring H) which satisfies equations (4) and (5) (equations (4), (5) and (6)).

In the second section of this paper a ring R is defined such that each linear graph L is associated with a unique element $f(L)$ of R, and it is shown that every W-function to G (V-function to H) can be expressed in the form $hf(L)$ where h is a homomorphism of R considered as an additive group (considered as a ring) into the group G (ring H), and that every such homomorphism is a W-function to G (V-function to H).

In the third section a V-function $Z(L)$ defined in terms of the subgraphs of L is studied; it is used in the next section in the proof of the following theorem.

THEOREM. *Let $(x_0, x_1, x_2, \ldots)$ be an infinite sequence of independent indeterminates over the ring I of rational integers. Then R is isomorphic with the ring of all polynomials over I in the x_i having no constant term.*

* The $p_1(L)$ of this paper is Whitney's 'nullity', and $p_0(L)$ is Whitney's P. The 'components' in this paper are Whitney's 'pieces': he uses the word 'component' with a different meaning. A footnote to Whitney's paper, dealing with some work of R. M. Forster, is particularly interesting with respect to the subject of the present paper.

Further there is a particular isomorphism in which the element of R corresponding to x_r is the element associated with a linear graph having just one 0-cell and just r 1-cells.

In the fifth section those W- and V-functions which are topological invariants of the complexes K are considered, and in the sixth section a particular V-function is applied to some colouring problems.

In the seventh section those 1-complexes K which admit of a simplicial dissection in which each 0-simplex is an end-point of not less than two and not more than three 1-simplexes are studied. A class of topologically invariant functions of these 1-complexes, one member of which is associated with a well-known colouring problem, is investigated, and it is shown that each of these functions has a unique extension as a topologically invariant W-function to all linear graphs.

2. The ring R

From the definitions of L'_A and L''_A it is evident that

$$\alpha_0(L) = \alpha_0(L'_A) \quad = \alpha_0(L''_A)+1 \tag{7}$$

and

$$\alpha_1(L) = \alpha_1(L''_A)+1 = \alpha_1(L'_A)+1. \tag{8}$$

We say that the link A is an *isthmus* if its suppression increases the number of components of a linear graph. Evidently

$$p_0(L) = p_0(L''_A) = p_0(L'_A) \quad \text{or} \quad p_0(L'_A)-1, \tag{9}$$

according as A is not or is an isthmus. Hence by (1)

$$p_1(L) = p_1(L''_A) = p_1(L'_A) \quad \text{or} \quad p_1(L'_A)+1, \tag{10}$$

according as A is or is not an isthmus.

We call the class of all linear graphs isomorphic with L the isomorphism class L^* of L. We also use clarendon type for isomorphism classes.

If $\mathbf{L}_1$ and $\mathbf{L}_2$ are any two isomorphism classes not necessarily distinct we can find L_1 in $\mathbf{L}_1$ and L_2 in $\mathbf{L}_2$ such that the product $L_1 L_2$ exists. All products formed in this way from $\mathbf{L}_1$ and $\mathbf{L}_2$ are clearly isomorphic. We call their isomorphism class $\mathbf{L}$ the *product* of $\mathbf{L}_1$ and $\mathbf{L}_2$ and write

$$\mathbf{L} = \mathbf{L}_1\mathbf{L}_2. \tag{11}$$

A *graphic form* is a linear form in the isomorphism classes $\mathbf{L}$ with integer coefficients of which only a finite number may be non-zero. We do not distinguish between an isomorphism class $\mathbf{L}$ and the graphic form in which the coefficient of $\mathbf{L}$ is unity and all the other coefficients are zero.

We define addition and multiplication for graphic forms by

$$\sum_i \lambda_i \mathbf{L}_i + \sum_i \mu_i \mathbf{L}_i = \sum_i (\lambda_i+\mu_i)\,\mathbf{L}_i \tag{12}$$

and

$$(\sum_i \lambda_i \mathbf{L}_i)\,(\sum_j \mu_j \mathbf{L}_j) = \sum_{i,j} (\lambda_i \mu_j)\,\mathbf{L}_i\mathbf{L}_j, \tag{13}$$

where the $\mathbf{L}_i$ are isomorphism classes and the λ_i are rational integers.

With these definitions the graphic forms are the elements of a commutative ring B. For the commutative, associative and distributive laws are evidently satisfied; and if $\mathbf{X} = \sum \lambda_i \mathbf{L}_i$ and $\mathbf{Y} = \sum_i \mu_i \mathbf{L}_i$ are any two graphic forms there is a unique graphic form $Z = \sum (\lambda_i - \mu_i)\,\mathbf{L}_i$ such that $\mathbf{Y}+\mathbf{Z} = \mathbf{X}$. We write $\mathbf{Z} = \mathbf{X}-\mathbf{Y}$.

If $\mathbf{X} = \sum_i \lambda_i \mathbf{L}_i$ is any graphic form and λ an integer, we denote by $\lambda\mathbf{X}$ the graphic form $\sum_i \lambda\lambda_i \mathbf{L}_i$. We also denote by $\mathbf{O}$ the graphic form whose coefficients are all zero.

If A is a link in a linear graph L we say that the graphic form $L^* - (L'_A)^* - (L''_A)^*$ is a *W-form*. Let W denote the set of all linear combinations of a finite number of W-forms taken with integer coefficients. Then W is a modul of B, for with $\mathbf{X}$ and $\mathbf{Y}$ it contains also $\mathbf{X} - \mathbf{Y}$.

Now if L_0 is any linear graph such that the product $L_0 L$ exists we have

$$(L_0 L)'_A = L_0 L'_A \quad \text{and} \quad (L_0 L)''_A = L_0 L''_A.$$

Therefore if $\mathbf{X}$ is any W-form and $\mathbf{L}$ any isomorphism class, then $\mathbf{LX}$ is also a W-form. Hence by (12) and (13) for any $\mathbf{Y} \in W$ and any $\mathbf{Z} \in B$, we have $\mathbf{YZ} \in W$. That is, W is an ideal of the commutative ring B. We denote the difference ring $B - W$ by R.

The elements of R are the cosets mod. W in B. If we denote the coset of $\mathbf{X}$ mod. W by $[\mathbf{X}]$, addition and multiplication in R satisfy

$$[\mathbf{X}] + [\mathbf{Y}] = [\mathbf{X} + \mathbf{Y}], \tag{14}$$

$$[\mathbf{X}][\mathbf{Y}] = [\mathbf{XY}]. \tag{15}$$

THEOREM I. *A single-valued function $W(L)$ on the set of all linear graphs L to an additive Abelian group G (commutative ring H) is a W-function (V-function) if and only if it is of the form $h[L^*]$ where h is a homomorphism of the additive group R (ring R) into G (H).*

Now the functions $W(L)$ which satisfy (4) depend only on the isomorphism classes. For such functions we write $W(\mathbf{L}) = W(L)$ where $\mathbf{L}$ is the isomorphism class of the linear graph L. $W(\mathbf{L})$ can now be extended to all graphic forms by writing

$$W\left(\sum_i \lambda_i \mathbf{L}_i\right) = \sum_i \lambda_i W(\mathbf{L}_i). \tag{16}$$

If $W(L)$ satisfies (6) we have also $W(\mathbf{L}_1\mathbf{L}_2) = W(\mathbf{L}_1)\, W(\mathbf{L}_2)$ for any two isomorphism classes $\mathbf{L}_1$ and $\mathbf{L}_2$, and therefore, by (13) and (16),

$$W(\mathbf{X}_1\mathbf{X}_2) = W(\mathbf{X}_1)\, W(\mathbf{X}_2), \tag{17}$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are any two graphic forms.

By (16) and (17) any single-valued function $W(L)$ satisfying equation (4) (equations (4) and (6)) is of the form $h_0 L^*$ where h_0 is a homomorphism of B considered as an additive group (considered as a ring) into G (H); and conversely it is evident that if h_0 is any such homomorphism, the function $h_0 L^*$ satisfies equation (4) (equations (4) and (6)).

$W(L)$ then satisfies (5) if and only if h_0 maps all W-forms and therefore all elements of W on to the zero element of G (H). This is equivalent to the condition that $h_0 L^*$ shall depend only on the coset $[L^*]$. The theorem now follows from (14) and (15).

Let y_r denote any linear graph having just one 0-cell and just r 1-cells (necessarily loops). Clearly all such linear graphs (for a fixed r) are isomorphic. We denote their isomorphism class by $\mathbf{y}_r$. We call the members of the $\mathbf{y}_r$ *elementary graphs*. Clearly

$$p_0(y_r) = 1, \tag{18}$$

$$p_1(y_r) = r. \tag{19}$$

THEOREM II. *If L is any linear graph then $[L^*]$ can be expressed as a polynomial $P[L^*] = P([L^*]; [\mathbf{y}_0], [\mathbf{y}_1], [\mathbf{y}_2], \ldots)$ in the $[\mathbf{y}_i]$ such that*

(i) *$P[L^*]$ has no constant term,*

(ii) *the coefficients of $P[L^*]$ are non-negative rational integers,*

(iii) *the degree of $P[L^*]$ is $\alpha_0(L)$,*

(iv) *$P[L^*]$ involves no suffix i greater than $p_1(L)$,*

and (v) *if L is connected and has no isthmus A such that for some component L_0 of L'_A, $p_1(L_0) = 0$; then $P[L^*]$ is of the form $[\mathbf{y}_p]+[\mathbf{Q}]$ where $p = p_1(L)$ and $[\mathbf{Q}]$ is a polynomial in those $[\mathbf{y}_i]$ for which i is less than p.*

The proof is by induction. We first observe that if $\alpha_1(L)$ is zero, then L is the product of $\alpha_0(L)$ elementary graphs each isomorphic with y_0. Hence by (15)

$$[L^*] = [\mathbf{y}_0]^{\alpha_0(L)},$$

and so the theorem is true for L.

Assume that the theorem is true for all connected linear graphs having fewer than some finite number n of 1-cells. Let L be any linear graph having just n 1-cells. If L is connected, then either $\alpha_0(L) = 1$, in which case

$$[L^*] = [\mathbf{y}_n],$$

and so the theorem is true for L, or else L contains a link A. In the second case we have

$$(L^* - (L'_A)^* - (L''_A)^*) \in W,$$

and therefore
$$[L^*] = [(L'_A)^*] + [(L''_A)^*]. \tag{20}$$

By (8) L'_A and L''_A have each fewer 1-cells than L and so by the inductive hypothesis the theorem is true for them. The propositions (i) to (iv) follow immediately for L from (20) with the help of (7) and (10).

Now suppose that L satisfies the conditions of (v). Then L''_A also satisfies these conditions since it is formed from L by identifying all the points and end-points of a link, a process which cannot alter the number of components of L'_B, where B is any link other than A of L. Hence by hypothesis

$$[(L''_A)^*] = [\mathbf{y}_p] + [\mathbf{Q}_p], \tag{21}$$

where p is $p_1(L)$, and $[\mathbf{Q}_p]$ denotes any polynomial (not always the same polynomial) in those $[\mathbf{y}_i]$ for which $i < p$.

We also have
$$[(L'_A)^*] = [\mathbf{Q}_p]. \tag{22}$$

This follows at once from (10) and (iv) when A is not an isthmus. If A is an isthmus, L'_A is of the form $L_0 L_1$. Since L satisfies the conditions of (v), $p_1(L_0), p_1(L_1) > 0$, and therefore since $p_1(L_0) + p_1(L_1) = p_1(L'_A)$ we have $p_1(L_0), p_1(L_1) < p_1(L'_A)$. Consequently $[(L'_A)^*] = [(L_0)^*][(L_1)^*] = [\mathbf{Q}_p]$ and (22) is still valid.

By (20), (21) and (22)
$$[L^*] = [\mathbf{y}_p] + [\mathbf{Q}_p].$$

This completes the proof that the theorem for connected linear graphs is true when $\alpha_1(L) = n$ if it is true for $\alpha_1(L) < n$. We have proved it for $\alpha_1(L) = 0$ and therefore it is true in general. If L is not connected we can obtain $P[L^*]$ satisfying the theorem by multiplying together the polynomials of its components.

COROLLARY. *Any element $[\mathbf{X}]$ of R can be expressed as a polynomial in the $[\mathbf{y}_i]$ with rational integer coefficients and no constant term.*

For $\mathbf{X}$ is a finite linear form in the $\mathbf{L}_i$ with integer coefficients.

3. Subgraphs

Let S denote any subgraph of a linear graph L. Let the number of components T of S such that $p_1(T) = r$ be $i_r(S)$. We define a function $Z(L)$ of L by

$$Z(L) = \sum_S \prod_r z_r^{i_r(S)}, \tag{23}$$

where the z_r are independent indeterminates over the ring I of rational integers. Although (23) involves a formal infinite product, yet for a given S only a finite number of the $i_r(S)$ can be non-zero and so, for each L, $Z(L)$ is a polynomial in the z_i.

Theorem III. *$Z(L)$ is a V-function.*

For first it is obvious that $Z(L)$ satisfies (4).

Secondly, if A is any link of L, then the subgraphs of L which do not contain A are simply the subgraphs of L'_A, and the subgraphs S of L which do contain A are in 1-1 correspondence with the subgraphs S''_A of L''_A. For, for such an S, S''_A is a subgraph of L''_A; and if S_1 is any subgraph of L''_A there is one and only one subgraph S of L having the same 1-cells as S_1 with the addition of A and therefore satisfying $S''_A = S$.

Further S''_A differs from S only in that a component T of S is replaced by T''_A; and, by (9) and (10), T''_A is connected and $p_1(T'') = p_1(T)$. Hence $i_r(S''_A) = i_r(S)$ for all r.

Hence by (23)
$$Z(L) = \sum_{S(L'_A)} \prod_r z_r^{i_r(S)} + \sum_{S(L''_A)} \prod_r z_r^{i_r(S)},$$
where $S(L'_A)$ for example denotes a subgraph S of L'_A. Therefore

$$Z(L) = Z(L'_A) + Z(L''_A), \tag{24}$$

so that $Z(L)$ satisfies (5).

Thirdly, for any product $L_1 L_2$ the subgraphs of $L_1 L_2$ are simply the products of the subgraphs S_1 of L_1 with the subgraphs S_2 of L_2. It is evident that

$$i_r(S_1 S_2) = i_r(S_1) + i_r(S_2),$$

and therefore
$$\begin{aligned} Z(L_1 L_2) &= \sum_{S_1, S_2} \prod_r z_r^{i_r(S_1)+i_r(S_2)} \\ &= \Big(\sum_{S_1} \prod_r z_r^{i_r(S_1)}\Big)\Big(\sum_{S_2} \prod_r z_r^{i_r(S_2)}\Big) = Z(L_1)\, Z(L_2). \end{aligned} \tag{25}$$

Thus $Z(L)$ satisfies (4), (5) and (6). That is, it is a V-function.

Theorem IV.
$$Z(y_r) = \sum_i \binom{r}{i} z_{r-i}. \tag{26}$$

For each subgraph of y_r has just one 0-cell (§ 2), and therefore just one component. Hence $Z(y_r)$ is a linear form in the z_r. The number of subgraphs S such that $p_1(S) = k$ is the number with $\alpha_1(S) = k$, by (19), and this is the number of ways of choosing k 1-cells out of r.

4. Structure of the ring R

Lemma.
$$\sum_{i=0}^{r} (-1)^i \binom{r}{i}\binom{i}{j} = (-1)^r \delta_{rj}. \tag{27}$$

This equality can be obtained by expanding $x^r = ((x-1)+1)^r$ in powers of $(x-1)$, expanding each of the terms in the resulting series in powers of x, and then equating coefficients.

THEOREM V. *R is isomorphic with the ring R_0 of all polynomials in the z_i with integer coefficients and no constant term.*

For by Theorem III $Z(L)$ is a V-function with values in R_0. Hence by Theorem I

$$Z(L) = h[L^*], \tag{28}$$

where h is a homomorphism of R into R_0.

Let $[\mathbf{t}_i]$ be the element of R defined by

$$[\mathbf{t}_i] = \sum_{j=0}^{i} (-1)^{i+j} \binom{i}{j} [\mathbf{y}_j]. \tag{29}$$

Then, by Theorem IV and the lemma,

$$h[\mathbf{t}_i] = \sum_{j=0}^{i} \sum_{s=0}^{j} (-1)^{i+j} \binom{i}{j} \binom{j}{s} z_s = z_i. \tag{30}$$

If we multiply (29) by $\binom{r}{i}$, sum from $i = 0$ to $i = r$, and use the lemma we find

$$[\mathbf{y}_r] = \sum_{i=0}^{r} \binom{r}{i} [\mathbf{t}_i]. \tag{31}$$

Hence by Theorem II, Corollary, any element $[\mathbf{X}]$ of R can be expressed as a polynomial in the $[\mathbf{t}_i]$ with integer coefficients and no constant term. Moreover this expression is unique; otherwise there would be a polynomial relationship between the $[\mathbf{t}_i]$, and therefore by (30) between the z_i, with integer coefficients, and this would contradict the definition of the z_i. It follows that h is an isomorphism of R on to R_0 (for every integer polynomial in the $[\mathbf{t}_i]$ is in R).

THEOREM VI. *Let $x_0, x_1, x_2, \ldots$ be an infinite sequence of connected linear graphs, and $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ the corresponding isomorphism classes, such that*

(i) $x_0 \cong y_0$,

(ii) $p_1(x_r) = r$,

and (iii) *x_r contains no isthmus A such that for some component L_0 of $(x_r)'_A$, $p_1(L_0) = 0$.*

Then any element $[\mathbf{X}]$ of R has a unique expression as a polynomial in the $[\mathbf{x}_i]$ with integer coefficients and no constant term.

By Theorem II (v) and equation (31) we have, for $r > 0$,

$$[\mathbf{x}_r] = [\mathbf{t}_r] + [\mathbf{S}_r], \tag{32}$$

where $[\mathbf{S}_r]$ is a polynomial in those $[\mathbf{t}_i]$ for which $i < r$. Hence

$$[\mathbf{t}_r] = [\mathbf{x}_r] + [\mathbf{U}_r], \tag{33}$$

where $[\mathbf{U}_r]$ is a polynomial in those $[\mathbf{x}_i]$ for which $i < r$. (If we assume this for $r < n$ it follows for $r = n$ by substitution in (32). Since $[\mathbf{x}_0] = [\mathbf{y}_0] = [\mathbf{t}_0]$ it is true for $r = 0$, and therefore it is true in general.) Clearly $[\mathbf{S}_r]$ and $[\mathbf{U}_r]$ have no constant terms.

By Theorem II, Corollary, and equations (31) and (33), $[\mathbf{X}]$ can be expressed as a polynomial without a constant term in the $[\mathbf{x}_i]$.

Suppose this expression not unique. Then there will be a polynomial relationship

$$P([\mathbf{x}_i]) = 0 \tag{34}$$

between the $[\mathbf{x}_i]$. Of the terms of non-zero coefficient in $P([\mathbf{x}_i])$ pick out the subset M_1 of those which involve the greatest suffix occurring in them raised to the highest power to which it occurs. Of this subset M_1 pick out the subset M_2 of terms involving the second greatest suffix appearing in M_1 raised to the highest power to which it occurs in M_1, and so on. This process must terminate in a subset M_k consisting of a single term

$$A[\mathbf{x}_i]^{\alpha(i)}[\mathbf{x}_j]^{\alpha(j)}\dots.$$

It is evident that if we substitute from (32) in (34), we shall obtain a polynomial relationship

$$Q([\mathbf{t}_i]) = 0$$

between the $[\mathbf{t}_i]$, in which the coefficient of $[\mathbf{t}_i]^{\alpha(i)}[\mathbf{t}_j]^{\alpha(j)}\dots$ is $A \neq 0$. But it was shown in the proof of Theorem V that there is no polynomial relationship between the $[\mathbf{t}_i]$.

This contradiction proves uniqueness and so completes the proof of the theorem.

5. Topologically invariant W-functions

Let A be a 1-cell of a linear graph L on a complex K. Let p be any point of A. We can obtain a new linear graph M on K from L by replacing A by the point p, taken as a 0-cell of M, and the two components of $A-p$ taken as 1-cells of M. We call this operation a subdivision of A by p.

Given any two linear graphs L_1, L_2 on the same K we can find a linear graph L_3 which can be obtained from either by suitable subdivisions. Such a linear graph is evidently obtained by taking as the set V of 0-cells the set of all points of K which are 0-cells either of L_1 or of L_2, and by taking as 1-cells the components of $K-V$.

We seek the condition that a W-function $W(L)$ shall be topologically invariant, i.e. depend only on K. By the above considerations a necessary and sufficient condition for this is that $W(L)$ shall be invariant under subdivision operations. (For then $W(L_1) = W(L_3) = W(L_2)$.)

Suppose therefore that A is any 1-cell of L, possibly a loop, and let M be obtained from L by subdividing A by a point p. Let us denote the new 1-cells by B and C. Then by (5) for any W-function $W(L)$

$$\begin{aligned} W(M) &= W(M'_B) + W(M''_B) \\ &= W((M'_B)'_C) + W((M'_B)''_C) + W(M''_B) \\ &= W(p\,.\,(M'_B)''_C) + W((M'_B)''_C) + W(M''_B). \end{aligned}$$

Here p is used to denote the linear graph which consists solely of the 0-cell p. It is isomorphic to y_0.

By making use of the obvious isomorphisms $M''_B \cong L$ and $(M'_B)''_C \cong L_0$, where L_0 is the linear graph derived from L by suppressing A, we obtain

$$W(M) - W(L) = W(y_0\,.\,L_0) + W(L_0),$$

Therefore

$$W(M) - W(L) = h([\mathbf{y}_0][L_0^*] + [L_0^*]), \tag{35}$$

where h is a homomorphism of R, regarded as an additive group, into an additive Abelian group G (Theorem I).

Let N denote the set of all elements of R which are of the form $[\mathbf{y}_0][\mathbf{X}] + [\mathbf{X}]$. Clearly N is an ideal of R. Let $\{\mathbf{X}\}$ denote that element of the difference ring $R-N$ which contains $[\mathbf{X}]$.

Theorem VII. *A function $W(L)$ on the set of all linear graphs L to the additive Abelian group G (commutative ring H) is a topologically invariant W-function (V-function) if* *and only if it is of the form* $k\{L^*\}$, *where* k *is a homomorphism of the additive group* $R-N$ (*ring* $R-N$) *into* G (H).

For in (35), by a proper choice of L, we can have any linear graph we please as L_0. It follows that the necessary and sufficient condition for the W-function $W(L)$ to be topologically invariant is that h shall map all elements of R of the form $[\mathbf{y}_0][L^*]+[L^*]$ and therefore all elements of N on to the zero of G. This proves the theorem for W-functions. The same argument applies to V-functions, except that h in (35) is then a homomorphism of R (as a ring) into the ring H.

THEOREM VIII. *Let* $x_0, x_1, x_2, \ldots$ *be as in the enunciation of Theorem VI.*

Then any element $\{\mathbf{X}\}$ *of* $R-N$ *has a unique expression as a polynomial in the* $\{\mathbf{x}_i\}$ $(i>0)$ *with integer coefficients.*

For we can obtain such an expression for $\{\mathbf{X}\}$ by replacing each $[\mathbf{x}_i]$ by the corresponding $\{\mathbf{x}_i\}$ in the expression for $[\mathbf{X}]$ in terms of the $[\mathbf{x}_i]$ whose existence is asserted in Theorem VI. Now for all $\{\mathbf{X}\}$, $\{\mathbf{X}\}+\{\mathbf{y}_0\}\{\mathbf{X}\} = \{\mathbf{O}\}$, and so $R-N$ has a unity element $-\{\mathbf{y}_0\} = -\{\mathbf{x}_0\}$ which we may denote by 1. Hence $\{\mathbf{x}_0\}$ is not an indeterminate over I, and we can regard our polynomial for $\{\mathbf{X}\}$ as a polynomial in those $\{\mathbf{x}_i\}$ for which $i>0$ (with perhaps a constant term).

If this expression for $\{\mathbf{X}\}$ is not unique then there will be a polynomial $\{\mathbf{P}\}$ in the $\{\mathbf{x}_i\}$ $(i>0)$ without a constant term such that

$$A\{\mathbf{x}_0\}+\{\mathbf{P}\} = \{\mathbf{O}\},$$

where A is some integer. Hence if $[\mathbf{P}]$ is the polynomial of the same form in the $[\mathbf{x}_i]$ we must have

$$A[\mathbf{x}_0]+[\mathbf{P}]+[\mathbf{X}_0]+[\mathbf{x}_0][\mathbf{X}_0] = [\mathbf{O}] \tag{36}$$

for some $[\mathbf{X}_0]$.

Equating coefficients of like powers of $[\mathbf{x}_0]$, as is permissible by Theorem VI, we see that $[\mathbf{X}_0]$ cannot involve $[\mathbf{x}_0]$, and hence that $A = -[\mathbf{X}_0] = [\mathbf{P}]$. Consequently $\{\mathbf{P}\}$ is a constant and therefore, by its definition, the zero polynomial in the $\{\mathbf{x}_i\}$. The theorem follows.

6. SOME COLOURING PROBLEMS

The homomorphism of the ring R_0 (see Theorem V) into the ring of polynomials in two independent indeterminates t and z by the correspondence $z_i \to tz^i$ transforms $Z(L)$ into

$$Q(L;t,z) = \sum_S t^{p_0(S)} z^{p_1(S)} \tag{37}$$

by (23). Since $Z(L)$ is of the form $h[L^*]$ where h is a homomorphism of R into R_0 (Theorems I and III). $Q(L;t,z)$ can be defined by a homomorphism of R into the ring of polynomials in t and z and is therefore a V-function (Theorem I).

The coefficient of $t^a z^b$, for fixed a, b, therefore satisfies (4) and (5) and so is a W-function. Writing $a=1, b=0$ we obtain the function of Example I of the Introduction. This function satisfies $W(L_1L_2)=0$ (by (37) since $p_0(S)$ is always positive) and so it can be regarded as a V-function with values in the ring constructed from the additive group of the rational integers by defining the 'product' of any two elements as 0.

$Q(L;t,z)$ has an interesting property which we call

THEOREM IX. *If L_1 and L_2 are connected dual linear graphs on the sphere then*

$$\frac{1}{t}Q(L_1;\, t, z) = \frac{1}{z}Q(L_2;\, z, t). \tag{38}$$

This follows from (37) as a consequence of the fact that there is a 1-1 correspondence $S \to S'$ between the subgraphs S of L_1 and the subgraphs S' of L_2 such that

$$p_0(S) = p_1(S')+1$$

and

$$p_1(S) = p_0(S')-1.$$

(S' is that subgraph of L_2 whose 1-cells are precisely those not dual to 1-cells of S.) For a proof of this proposition reference may be made to the paper 'Non-separable and planar graphs' by Hassler Whitney (*Trans. American Math. Soc.* 34 (1932), 339–62).

We go on to consider two kinds of colourings of a linear graph, which we distinguish as *α-colourings* and *β-colourings*. An α-colouring of L of degree λ is a single-valued function on the set of 0-cells of L to a fixed set H the number of whose elements is λ.

If f is an α-colouring let $\phi(f)$ denote the number of 1-cells A of L such that f associates all the end-points of A with the same element of H (e.g. every loop has this property). We say that any subgraph of L all of whose 1-cells have this property for f is *associated* with f. We use the symbol $S(f)$ to denote a subgraph associated with a given f, and $f(S)$ to denote any α-colouring with which a given S is associated.

THEOREM X. *Let $J(L;\, \lambda, \phi)$ be the number of α-colourings f of L of degree λ for which $\phi(f)$ has the value ϕ. Then the following identity is true.*

$$\sum_{\phi} J(L;\, \lambda, \phi)\, x^{\phi} = (x-1)^{\alpha_0(L)}\, Q\left(L;\, \frac{\lambda}{x-1}, x-1\right) \tag{39}$$

where x is an indeterminate over I.

For, by (37) and (1), the right-hand side is

$$\begin{aligned}(x-1)^{\alpha_0(L)} \sum_{S} \lambda^{p_0(S)}(x-1)^{p_1(S)-p_0(S)} &= \sum_{S}(x-1)^{\alpha_1(S)}\lambda^{p_0(S)} \\ &= \sum_{S}\left((x-1)^{\alpha_1(S)} \sum_{f(S)} 1\right);\end{aligned}$$

for the α-colourings associated with S are precisely those which map all the 0-cells in the same component of S on to the same element of H. This last expression equals

$$\sum_{f}\sum_{S(f)}(x-1)^{\alpha_1(S)} = \sum_{f} x^{\phi(f)}$$

since the number of subgraphs associated with f and having just $\alpha_1(S)$ 1-cells is the number of ways of choosing $\alpha_1(S)$ 1-cells out of $\phi(f)$. This completes the proof of the theorem.

If we write $x = 0$ in (39) we find that $(-1)^{\alpha_0(L)} J(L;\, \lambda, 0)$, which is Example II of the Introduction, is the V-function $Q(L;\, -\lambda, -1)$. We thus obtain the well-known result*

$$J(L;\, \lambda, 0) = \sum_{S}(-1)^{\alpha_1(S)}\lambda^{p_0(S)}.$$

* Hassler Whitney, 'A logical expansion in mathematics', *Bull. American Math. Soc.* 38 (1932), 572–9.

If we orient the 1-cells of L and adopt the convention that the boundary of an oriented loop vanishes, we can define 1-cycles on L with coefficients in some fixed additive Abelian group G of finite order λ. The number* of such 1-cycles on L will be $\lambda^{p_1(L)}$. We call them *β-colourings* of L with respect to G.

Let $E(L;\, G, \psi)$ be the number of such 1-cycles for which just ψ of the 1-cells have coefficient zero. Let g_G be any β-colouring with respect to G of L and let $\psi(g_G)$ be the number of its zero coefficients†. We say that a subgraph S of L is associated with g_G if every 1-cell of L not in S is assigned the zero element of G as its coefficient in g_G. We use the symbol $S(g_G)$ to denote a subgraph of L associated with g_G and $g_G(S)$ to denote a β-colouring with which a given subgraph is associated. Clearly the number of β-colourings associated with a given S is the number of β-colourings of S, which is $\lambda^{p_1(S)}$.

THEOREM XI. *If x is an indeterminate over I then*

$$\sum_{\psi} E(L;\, G, \psi)\, x^{\psi} = (x-1)^{\alpha_1(L)-\alpha_0(L)}\, Q\left(L;\, x-1, \frac{\lambda}{x-1}\right). \tag{40}$$

For, by (37) and (1), the right-hand side is

$$\begin{aligned}(x-1)^{\alpha_1(L)-\alpha_0(L)} \sum_{S} (x-1)^{p_0(S)-p_1(S)} \lambda^{p_1(S)} &= \sum_{S} \left((x-1)^{\alpha_1(L)-\alpha_1(S)} \sum_{g_G(S)} 1\right) \\ &= \sum_{g_G} \sum_{S(g_G)} (x-1)^{\alpha_1(L)-\alpha_1(S)} = \sum_{g_G} x^{\psi(g_G)};\end{aligned}$$

for the number of subgraphs of L associated with g_G and having just $\alpha_1(L) - \psi(g_G) + r$ 1-cells is the number of ways of choosing r 1-cells out of the $\psi(g_G)$ which have zero coefficient in g_G.

COROLLARY. *$E(L;\, G, \psi)$ is the same for all additive Abelian groups G of the same order λ.*

If we write $x = 0$ in (40) we find that $(-1)^{\alpha_1(L)-\alpha_0(L)}\, E(L;\, G, 0)$, which is Example III of the Introduction, is the V-function $Q(L;\, -1, -\lambda)$. It takes the value -1 when L is y_0 and therefore corresponds to a homomorphism of R into the ring of rational integers which maps N into 0. It is therefore, by the preceding section, topologically invariant.

If L_1 and L_2 are dual linear graphs on the sphere, the β-colourings of L_1 are closely connected with the α-colourings of L_2. In fact a 1-cycle g_G bounds on the sphere and any 2-chain which it bounds on the map defined by L_1 has a dual 0-chain which is an α-colouring f_λ of L_2 such that $\phi(f_\lambda) = \psi(g_G)$.

There is also a relationship between the α-colourings and the β-colourings of the same linear graph L expressed by the following identity in x

$$(x-1)^{\alpha_1(L)} \sum_{\psi} \left(E(L;\, G, \psi) \left(\frac{\lambda}{x-1} + 1\right)^{\psi}\right) = \lambda^{\alpha_1(L)-\alpha_0(L)} \sum_{\phi} J(L;\, \lambda, \phi)\, x^{\phi}. \tag{41}$$

This is obtained by writing $\lambda/(x-1)$ for $(x-1)$ in (40) and then eliminating the function Q by means of (39).

* See Lefschetz, *Algebraic Topology* (Amer. Math. Soc. Colloquium Publications, vol. 27), p. 106.

† It may be mentioned that for graphs on the sphere a β-colouring is essentially equivalent to a colouring of the regions of the map defined by a graph in λ colours. The colours can be represented by elements of G and so the colouring can be represented by a 2-chain on the map with coefficients in G. A β-colouring is simply the boundary of such a 2-chain. The number of 1-cells incident with two regions of the same colour (or incident with only one region) in a given colouring is given by the number $\psi(g_G)$ where g_G is the corresponding β-colouring.

7. Cubical networks

We define a *cubical network* as a 1-complex for which there exists a finite simplicial dissection in which each 0-simplex is incident with not less than two, and not more than three 1-simplexes. Clearly any other simplicial dissection of such a complex will have the same property. The 0-simplexes which are each incident with three 1-simplexes we call *nodes*. The set of nodes is evidently independent of the particular simplicial dissection taken.

A component of a cubical network which does not contain a node is evidently a simple closed curve, and if a component does contain nodes then the remainder of it

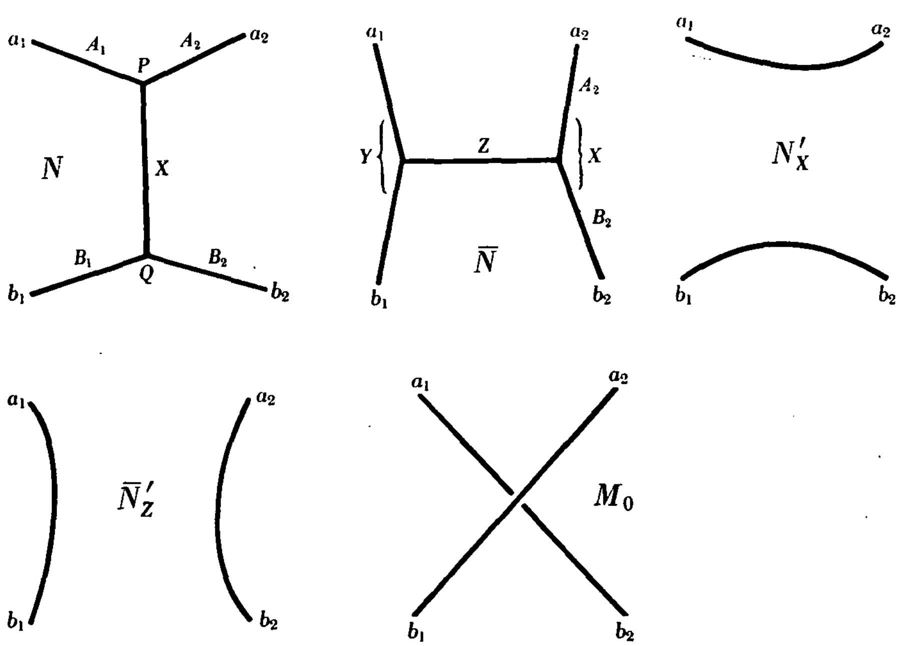

Fig. 1

must consist of a number of non-intersecting open arcs whose end-points are nodes of the component. We call these open arcs the *arcs* of the cubical network.

The number of nodes in a cubical network N is clearly two-thirds of the number of arcs of N. It is therefore even.

Let X be an arc having distinct end-points P and Q in a cubical network N. In a simplicial dissection of N let A_1, A_2 be those 1-simplexes incident with P, and B_1, B_2 those 1-simplexes incident with Q, which are not in X. Let a_1, a_2, b_1, b_2 be the other end-points of A_1, A_2, B_1, B_2 respectively. By suitable subdivisions of a given simplicial dissection we can always arrange that a_1, b_1, a_2, b_2 are distinct points and not nodes of N.

Other cubical networks can be obtained from N by replacing X, A_1, A_2, B_1, B_2, P and Q by other systems of simplexes (see Fig. 1). If for example we suppress A_1 and B_1, introduce a new arc Y joining a_1 to b_1 and then introduce an arc Z joining a point in Y

to a point in X, we obtain $\bar{N}$. We call this process a Λ-*operation* on N. If N_1 can be obtained from N by a finite sequence of Λ-operations we say that N and N_1 are Λ-*equivalent*. In such a case it is clear that N_1 has the same number of nodes as N and that if N is connected, so is N_1.

By suppressing X in N we obtain N'_X, and by suppressing Z in $\bar{N}$ we obtain $\bar{N}'_Z$. We define an *F-function* as a single-valued topologically invariant function on the set of all cubical networks to an additive Abelian group G or commutative ring H which satisfies the general law

$$F(N)-F(N'_X) = F(\bar{N})-F(\bar{N}'_Z). \tag{42}$$

THEOREM XII. *If $W(L)$ is a topologically invariant W-function, and $F(N)$ is the value of $W(L)$ for any linear graph on the cubical network N, then $F(N)$ is an F-function.*

For let N_0 be the 1-complex obtained from N by identifying all the points of the closure of X, and let L_0 be any linear graph on N_0 (clearly such exist). N_0 is evidently homoeomorphic to the 1-complex obtained from $\bar{N}$ by identifying all the points of the closure of Z. Since $W(L)$ is topologically invariant it follows from (5) that

$$F(N)-F(N'_X) = W(L_0) = F(\bar{N})-F(\bar{N}'_Z),$$

which proves the theorem.

A trivial example of an F-function is $F(N) = x^{n(N)}$ where x is an arbitrary real or complex number and $n(N)$ is one-half of the number of nodes of N. This function also satisfies

$$F(N_1 \cup N_2) = F(N_1)\,F(N_2), \tag{43}$$

where N_1 and N_2 are any two disjoint cubical networks and $N_1 \cup N_2$ is their union.

Other F-functions may be obtained as follows. We define a *subnetwork* of N as a 1-complex which is the union of all the nodes of N and some subset of the arcs and nodeless components of N, such that each node of N is an end-point of at least one arc of the subset. If the number of arcs of a subnetwork T which have a given node v of N as an end-point (arcs which are loops being counted twice) is odd, we say that v is an *odd node* of T. The number of odd nodes of T is even, for it is congruent mod. 2 to the number of end-points of arcs of T (a loop being regarded as having two end-points, though they happen to coincide). Let $k(T)$ be one-half the number of odd nodes of T. Let $\pi_k(N)$ be the number of subnetworks of N for which $k(T) = k$. As an example a cubical network J which consists of a single simple closed curve has just two subnetworks—J itself and the null complex—and so $\pi_0(J) = 2$ and $\pi_i(J) = 0$ $(i > 0)$.

Let M be the 1-complex obtained from the cubical network N of Fig. 1 by suppressing X, A_1, A_2, B_1 and B_2. Let T be any subnetwork of N, N'_X, $\bar{N}$ or $\bar{N}'_Z$, and let T_0 be its intersection with M (which is contained in each of these four complexes). If we are told which of a_1, a_2, b_1, b_2 are contained in T_0 it is easy to determine for each of the four cubical networks how many subnetworks there are which agree with T_0 in M, and how many of these have 0 (or 1, or 2) odd nodes outside T_0. A consideration of the possible cases will show

$$\pi_k(N)+\pi_k(N'_X) = \pi_k(\bar{N})+\pi_k(\bar{N}'_Z), \tag{44}$$

whence $(-1)^{n(N)}\pi_k(N)$ satisfies (42) and is thus an F-function.

If therefore we define a polynomial $D(N; x)$ by

$$D(N; x) = \sum_k \pi_k(N)\,x^k$$

then $(-1)^{n(N)}D(N; x)$ will be an F-function. Further, by an argument analogous to the proof of (25) this F-function satisfies (43).

If N has no nodeless component, $\pi_0(N) = D(N; 0)$ is by its definition the number of solutions of Petersen's problem* for N.

We define a *Hamiltonian circuit* of N as a subnetwork of N which is connected and has no odd nodes. It is easily verified that the residue mod. 2 of the number of Hamiltonian circuits of N satisfies (42), so this also is an F-function.

Let γ_{i+1} $(i \geqslant 1)$ be a cubical network with just $2i$ nodes $a_1, a_2, a_3, \ldots, a_{2i}$, having just one arc linking each pair of nodes a_r, a_{r+1} for which r is odd, having just two arcs linking each pair of nodes a_r, a_{r+1} for which r is even, and having two arcs which are loops the end-points of one coinciding in a_1 and those of the other in a_{2i}. The nodes and arcs define a linear graph which we also denote by γ_{i+1}.

THEOREM XIII. *Any connected cubical network N of $2n$ nodes $(n > 0)$ is Λ-equivalent to a homoeomorph of γ_{n+1}.*

For first, if N, not being homoeomorphic to γ_{n+1}, contains a simple closed curve K of $k > 1$ arcs, then N is Λ-equivalent to a cubical network N_1 containing a simple closed curve of $k-1$ arcs. For we can suppose that K contains the arc X (Fig. 1) and also a_1 and b_1. Then $\bar{N}$ clearly has the property desired. It follows that by a sequence of Λ-operations we can convert N into a cubical network having a loop.

Let δ_r be the 1-complex derived from γ_{r+1} $(r > 0)$ by suppressing the loop on a_{2r}. If part of a cubical network M meeting the rest of M only in a single node is homoeomorphic with δ_r, we call it a *frond* of M of *degree* r, and say that the node corresponding to a_{2r} is the *base* of the frond. The above argument showed that N is Λ-equivalent to a cubical network N_2 having a frond f (of degree r say).

Secondly either N_2 contains a simple closed curve passing through the base of f, or it is Λ-equivalent to a cubical network having a frond of degree at least r with a simple closed curve through its base. For if the base c_0 of f is not on such a curve there will be a sequence $c_0, c_1, c_2, c_3, \ldots, c_s$ of minimum length such that consecutive nodes c_i, c_{i+1} are linked by an arc C_i, and such that c_s is on a simple closed curve K_1 in N_2. Otherwise we could extend the sequence $c_0, c_1, c_2, \ldots$ indefinitely in such a way that C_i differed from C_{i+1} for each i without repetitions, which is absurd since N_2 has only a finite number of nodes. By Λ-operations on $C_0, C_1, \ldots$ in turn it is possible to transfer the frond to a base on a simple closed curve without altering its degree.

Now at this stage the simple closed curve through the base of the frond may be a loop, in which case N has been transformed into a γ_i-homoeomorph, and $i = n+1$ since connexion and number of nodes are invariant under Λ-operations; or it may contain just two arcs in which case N_2 has been transformed into a cubical network having a frond of degree exceeding r; or it can be reduced to a curve of just two arcs by a sequence of Λ-operations on those of its arcs not meeting the base of the frond. Hence if N_2 is not homoeomorphic with γ_{n+1} it can be transformed into a cubical network with a frond of degree greater than r. A finite number of such transformations will therefore change it into a homoeomorph of γ_{n+1}.

* Dénes König, *Theorie der Endlichen und unendlichen Graphen* (Leipzig, 1936), p. 186.

Theorem XIV. *Let $F(N)$ be any F-function. Then there is a unique topologically invariant W-function $W(L)$ such that $W(L) = F(N)$ whenever L is a linear graph on N.*

For the linear graphs γ_{i+1} may be taken as the linear graphs x_{i+1} of Theorem VI. If we make the definitions $\gamma_0 = y_0$ and $\gamma_1 = y_1$ then the γ_i clearly satisfy the conditions of Theorem VI, and so by Theorem VIII $\{L^*\}$ has a unique expression as a polynomial in the $\{\gamma_i\}$. Hence there is a unique topologically invariant W-function $W(L)$ which is equal to $F(N)$ whenever N is a product of γ_i and L is on N. By Theorem XII there is a unique F-function $F_1(N)$ such that $W(L) = F_1(N)$ whenever L is on N.

But if the value of an F-function is given for every product of γ_i, then it is determined for all N. For by (42) if it is known for all N such that $n(N) = p$ and for one cubical network M such that $n(M) = p+1$, then it is determined for any cubical network M_1 Λ-equivalent to a homoeomorph of M. By applying Theorem XIII to each component having a node we see that every cubical network is Λ-equivalent to a homoeomorph of a product of γ_i and so the required result follows by induction. Since $F(N) = F_1(N)$ whenever N is a product of γ_i it follows that $F(N) = F_1(N)$ for every cubical network N. This proves the theorem.

Corollary. *For an F-function satisfying* (43) '*W-function*' *can be replaced by* '*V-function*' *in the above argument.*

As an example we mention an application of the above theory to the problem of functions obeying the law

$$f(\overline{N}) = f(N'_X) + f(M_0) \tag{45}$$

(see Fig. 1).

By eliminating $f(M_0)$ from two equations of the form (45) it is easy to show that $f(N)$ is an F-function multiplied by $(-1)^{n(N)}$. Hence it is fixed when its values for the products of the γ_i are given. But by applying (45) to these products we can show that for them $f(N) = 2^{n(N)}A$ where A is a constant. Since $2^{n(N)}A$ is obviously a solution of (45) it follows that it is the general solution.

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