We have a 3-(v, k, λ) design $\mathscr{D}$ such that the derived 2-design $\mathscr{D}_p$ is symmetric, where $p \in \mathscr{P}$, the point set. (I assume this for every point p; I'm sure that's what they intended, by their wording and also because it's necessary.) We're asked to prove several things, falling into five parts.

(1) An equation.

Proposition 1. $(k-1)(k-2) = (v-2)\lambda$.

The derived design is a 2-$(v-1, k-1, \lambda)$ design (easy to prove, and it's the Corollary to Theorem 19.3). By symmetry, $r_p = k_p - k - 1$. By Equation (19.4), $r_p(k_p - 1) = (v_p - 1)\lambda$ for a 2-design. By substitution $(k-1)(k-2) = (v-2)\lambda$. ■

(2) Intersections.

Proposition 2. *Two blocks that intersect have $\lambda + 1$ common points.*

Suppose $B_1, B_2 \in \mathscr{B}$ do intersect; let p be a common point. Then $B_1 \setminus p, B_2 \setminus p$ are blocks in the 2-$(v-1, k-1, \lambda)$ design $\mathscr{D}_p$, so they each have $k_p = k - 1$ points. They are blocks of a symmetric design, so by Theorem 19.9 the dual design is also a 2-$(v-1, k-1, \lambda)$ design, so $|(B_1 \setminus p) \cap (B_2 \setminus p)| = \lambda$. It follows that $|B_1 \cap B_2| = \lambda + 1$. ■

I know Theorem 19.9 is later. I think everyone (including me) managed to prove this in a different way without going ahead in the book, so I'm showing the elegant proof.

(3) Residual.

Proposition 3. *The residual system $\mathscr{D}^B$ of any block B is a 2-design.*

Here $\mathscr{P}^B = \mathscr{P} \setminus B$ and $\mathscr{B}^B = \{B' \in \mathscr{B} : B' \cap B = \varnothing\}$. Thus, $\mathscr{D}^B$ has $v^B = v - k$, $k^B = k$, and we only have to establish the existence of $\lambda^B =$ the number of blocks of $\mathscr{D}^B$ on two points $x, y \notin B$. These are blocks $B' \in \mathscr{B}$ that are disjoint from B, which we calculate by first counting the blocks $B' \ni x, y$ of $\mathscr{D}$ that do intersect B; let this number be c.

This is a bit complicated! Consider only blocks B' on x, y. The number of pairs (B', p) with $p \in B' \cap B$ is the sum over all such B' of the number of points $p \in B' \cap B$, which is $\lambda + 1$ by (2). This number equals $c(\lambda + 1)$. Counting pairs another way, for each $p \in B$ there are λ blocks containing x, y, p; thus the number of pairs is $k\lambda$. It follows that

$$c = \frac{k\lambda}{\lambda + 1}. \tag{a}$$

Now $\lambda^B = b_2 - c$, because the total number of blocks on x, y is b_2 of Theorem 19.3. So, recalling that $t^B = 2$, we get

$$\lambda^B = \lambda\frac{v-1}{k-1} - \lambda\frac{k}{\lambda+1} = \lambda\Big[\frac{v-1}{k-1} - \frac{k}{\lambda+1}\Big]. \tag{b}$$

Since this is independent of the choice of x, y, we have a 2-design $\mathscr{D}^B$. ■

From Proposition 1 we get $\lambda^B > 0$; I omit the calculation.

Equation (a) implies

$$\lambda + 1 | k$$

and then Equation (b) implies

$$k - 1 | v - 1.$$

(4) Fisher-y.

This is where it gets sticky. I'm stuck! (so far).

I assume $k \geq 3$, since for a 3-design with $k < 3$, $\lambda = 0$ and it's trivial, boring, and perhaps slightly disgusting.

In order to apply Fisher's inequality to $\mathscr{D}^B$ as hinted, we need $b^B > 1$. Thus, there are two cases. Actually, there are three cases.

Case 0: $v < 2k$.
Case 1: $v = 2k$, $b^B = 1$.
Case 2: $v > 2k$, $b^B \geq v^B = v - k$ (by Fisher).

The problem here is to show the equations for k. TO BE DONE

First, we have to calculate b^B.

(5) Examples.

Case 0: Since $v^B < k^B$, $b^B = 0$. I don't think we're supposed to take this seriously. If we did, we could calculate b^B and show that this case occurs if and only if $v = k + 1$ or $(\lambda + 1)(v - 1) = k(k - 2)$. I haven't succeeded in classifying these examples. For the first type, the complementary design has $\bar{k} = 1$, which seems ridiculous, and I don't know what to make of it. For the second type, $(\lambda+1)v = (k-1)^2$ and since $\lambda+1|k$, this looks improbable, but I don't know.

Case 1: The hint implies this should be a Hadamard-type design. That remains TO BE DEVELOPED.

Case 2: TO BE DONE.