Lemma for Erdős–Ko–Rado Theorem 6.4

We assume $0 < k < n/2$. The intersecting family of k-sets contained in $[n]$ is $\mathcal{A}$. There was a missing part of the proof because Case 2 was too complicated. I think it really is complicated, but maybe there is an easier proof.

Lemma 1 (Lemma 6.4A)**.** *The number of k-sets of $\mathcal{A}$ that are intervals in a circular ordering $\mathcal{F}^\pi$ is at most k.*

Proof. We use $\mathcal{F}$ and assume one of the sets is $A_0 = [k]$. Then any other set $S \in \mathcal{A} \cap \mathcal{F}$ has $\max S \geq 1$ and $\min S \leq k$ in order to overlap A_0.

Case 1: The sets don't wrap around; i.e., there is an $i \in [n]$ such that none of the sets contains both i and $i+1$. Choose the biggest possible i; then we can assume $i = n$ by relabelling and no set S will contain n, since $\min S \geq 1$ and $\max S \leq 2k-1 < n$. So the only possible sets are $A_0 + i$ for $i = 0, 1, \ldots, k-1$.

Case 2: The sets wrap around. Then there have to be sets $B, C \in \mathcal{A} \cap \mathcal{F}$ such that $n \in B$, $k+1 \in C$, and $A_0 \cap B \cap C = \varnothing$. (The B's and C's are distinct because no k-interval can contain both n and $k+1$ and also intersect A_0.)

Each B is characterized by $j = \max B$. Among all B's let $j_{\max}$ and $j_{\min}$ be the largest and smallest values of j. Each B has $j_{\min} \leq \max B \leq j_{\max}$, so the number of B's is at most $j_{\max} - j_{\min} + 1$. For later use, let $B_{\min}$ be the B with $j = j_{\min}$.

Each C is characterized by $h = \min C$. Among all C's let $h_{\max}$ and $h_{\min}$ be the largest and smallest values of h. Each C has $h_{\min} \leq \min C \leq h_{\max}$, so the number of C's is at most $h_{\max} - h_{\min} + 1$. For later use, let $C_{\max}$ be the C with $h = h_{\max}$.

In order for all $B \cap C$ to be nonempty, we must have $h_{\min} + k - 1 \geq j_{\max} - k + 1 + n$, because $\max C_{\min} = h_{\min} + k - 1$ and (adjusted by adding n) $\min B_{\max} = j_{\max} - (k-1) + n$. Thus, $h_{\min} - j_{\max} \geq n + 2 - 2k$.

A k-interval $A \in \mathcal{A} \cap \mathcal{F}$ that is not a B or a C has the property that the nonempty intersections $A \cap B, A \cap C \subseteq A_0$. Since $A \cap C_{\max} \neq \varnothing$, we have $i = \max A \geq h_{\max}$. Since $A \cap B_{\min} \neq \varnothing$, we have $i - k + 1 = \min A \leq j_{\min}$. Thus, $h_{\max} \leq i \leq j_{\min} + k - 1$. It follows that the number of A's is at most $(j_{\min} + k - 1) - h_{\max} + 1$; this includes A_0, for which $i = k$.

Every k-interval that belongs to $\mathcal{A}$ is an A, a B, or a C. Therefore, the total number of such k-intervals is at most

$$
\begin{aligned}
&\big[(j_{\min} + k - 1) - h_{\max} + 1\big] + \big[j_{\max} - j_{\min} + 1\big] + \big[h_{\max} - h_{\min} + 1\big] \\
&= k + 2 - (h_{\min} - j_{\max}) \\
&\leq k + 2 - (n + 2 - 2k) = 3k - n \leq k
\end{aligned}
$$

since $3k \leq 3n/2$. □

Case 1 was the easy part, though here I'm making it a little less obvious in order to simplify the proof without a diagram.

Note that the upper bound in Case 2 is attainable only when $k = n/2$.

Example 1. To show the upper bound is attainable in Case 2 when $n = 2k$, consider $k = 3$. Let $\mathcal{A} \cap \mathcal{F} = \{123, 345, 561\}$. This has k sets in it. It should be easy to generalize to any $k > 0$.