

NOTE

A Shorter, Simpler, Stronger Proof of the Meshalkin–Hochberg–Hirsch Bounds on Componentwise Antichains

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Meshalkin's theorem states that a class of ordered p -partitions of an n -set has at most $\max_{a_1, \dots, a_p} \binom{n}{a_1, \dots, a_p}$ members if for each k the k th parts form an antichain. We give a new proof of this and the corresponding LYM inequality due to Hochberg and Hirsch, which is simpler and more general than previous proofs. It extends to a common generalization of Meshalkin's theorem and Erdős's theorem about r -chain-free set families. © 2002 Elsevier Science (USA)

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An *antichain of sets* is a class of sets of which none contains another. Sperner [8] bounded the size of an antichain \mathcal{A} of subsets of an n -element set S :

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor},$$

with equality if $\mathcal{A} = \mathcal{P}_{\lfloor n/2 \rfloor}(S)$ or $\mathcal{P}_{\lceil n/2 \rceil}(S)$, where by $\mathcal{P}_k(S)$ we mean the class of k -element subsets of S . Subsequently, Lubell [5], Yamamoto [9], and Meshalkin [6] independently obtained a stronger result (of which Bollobás independently proved a generalization [2]): any antichain \mathcal{A} satisfies

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1,$$

and equality holds if $\mathcal{A} = \mathcal{P}_{\lfloor n/2 \rfloor}(S)$ or $\mathcal{P}_{\lceil n/2 \rceil}(S)$.

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We give a very short proof of a considerable generalization of these results. A class \mathcal{A} of subsets of S is *r-chain-free* if \mathcal{A} contains no chain of length r .² (A *chain* is a class of mutually comparable sets, that is, $T \subset T' \subset \dots \subset T^{(l)}$. Its *length* is l .) A *weak composition of S into p parts* is an ordered p -tuple $A = (A_1, \dots, A_p)$ such that the A_k are pairwise disjoint subsets of S and their union is S . We call A_k the *kth part* of A . A part A_k may be void (hence the word “weak”). If $\mathcal{M} = \{A^1, \dots, A^m\}$ is a class of weak compositions of S into p parts, we write $\mathcal{M}_k = \{A_k^i\}_{i=1}^m$ for the class of distinct *kth parts* of members of \mathcal{M} . A multinomial coefficient of the form $\binom{n}{a_1, \dots, a_p}$ is called a *p-multinomial coefficient for n*. Our result is:

THEOREM. *Let $n \geq 0, r \geq 1, p \geq 2$, and let S be an n -element set. Suppose \mathcal{M} is a class of weak compositions of S into p parts such that, for each $k < p$, \mathcal{M}_k is r -chain-free. Then*

$$(a) \sum_{A \in \mathcal{M}} \frac{1}{\binom{n}{|A_1|, \dots, |A_p|}} \leq r^{p-1}, \text{ and}$$

(b) $|\mathcal{M}|$ is bounded by the sum of the r^{p-1} largest p -multinomial coefficients for n .

The number of p -multinomial coefficients for n is $\binom{n+p-1}{p-1}$; if r^{p-1} exceeds this we extend the sequence of coefficients with zeros.

Our theorem is a common generalization of results of Meshalkin and Erdős. The case $r = 1$ of part (b) (with the added assumption that every \mathcal{M}_k is an antichain) is the relatively neglected theorem of Meshalkin [6]; later Hochberg and Hirsch [4] found (a) for this case, which implies (b). Our extension to $r > 1$ is inspired by the case $p = 2$, which is equivalent to Erdős’s theorem [3] that for an r -chain-free family \mathcal{A} of subsets of S , $|\mathcal{A}|$ is bounded by the sum of the r largest binomial coefficients $\binom{n}{k}, 0 \leq k \leq n$, and its LYM companion due to Rota and Harper [7]. We need the latter for our theorem; we sketch its proof for completeness’ sake.

LEMMA (Rota and Harper [7, p. 198, (*)]). *For an r -chain-free family \mathcal{A} of subsets of S ,*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq r.$$

Proof. Each of the $n!$ maximal chains in $\mathcal{P}(S)$ contains at most r members of \mathcal{A} . On the other hand, there are $|A|!(n - |A|)!$ maximal chains

² An r -chain-free family has been called an “ r -family” or “ k -family”, depending on the name of the forbidden length, but we think it is time for a distinctive name.

containing $A \in \mathcal{P}(S)$. Now count, giving

$$\sum_{A \in \mathcal{A}} |A|!(n - |A|)! \leq rn!$$

The lemma follows. ■

Proof of the Theorem. Our proof of the whole theorem is simpler than the original proofs of the case $r = 1$. The proof of (a) here (which is different from the more complicated although equally short proof by Hochberg and Hirsch) is inspired by the beginning of Meshalkin’s proof of (b) for $r = 1$. We proceed, as did Meshalkin, by induction on p . The case $p = 2$ is equivalent to the lemma because if (A_1, A_2) is a weak composition of S , then $A_2 = S \setminus A_1$. Suppose then that $p > 2$ and (a) is true for $p - 1$. Let $\mathcal{M}(F) = \{(A_2, \dots, A_p) : (F, A_2, \dots, A_p) \in \mathcal{M}\}$. Since $\mathcal{M}(F)_k \subseteq \mathcal{M}_{k+1}$, $\mathcal{M}(F)_k$ is r -chain-free for $k < p - 1$. Thus,

$$\begin{aligned} \sum_{A \in \mathcal{M}} \frac{1}{\binom{n}{|A_1|, \dots, |A_p|}} &= \sum_{A \in \mathcal{M}} \frac{1}{\binom{n}{|A_1|}} \frac{1}{\binom{n - |A_1|}{|A_2|, \dots, |A_p|}} \\ &= \sum_{F \in \mathcal{M}_1} \frac{1}{\binom{n}{|F|}} \sum_{A' \in \mathcal{M}(F)} \frac{1}{\binom{n - |F|}{|A_2|, \dots, |A_p|}}, \end{aligned}$$

where $A' = (A_2, \dots, A_p)$,

$$\leq \sum_{F \in \mathcal{M}_1} \frac{1}{\binom{n}{|F|}} r^{p-2}$$

by the induction hypothesis,

$$\leq r r^{p-2}$$

by the lemma. This proves (a).

To deduce (b), write the p -multinomial coefficients for n in any weakly decreasing order as M_1, M_2, \dots , extended by 0’s as necessary to a sequence of length r^{p-1} . In the left-hand side of (a), replace each of the M_1 terms with largest denominators by $1/M_1$. Their sum is now 1. Amongst the remaining terms all denominators are at most M_2 ; replace the M_2 of them with the largest denominators by $1/M_2$. Now their sum is 1. Continue in this fashion. The number of terms could be less than $T = M_1 + \dots + M_{r, p-1}$; in that case, $|\mathcal{M}| < T$. Otherwise, after r^{p-1} steps we have replaced T terms and have on the left-hand side of (a) a sum equal to r^{p-1} plus any further terms. As the

total is no more than r^{p-1} , there cannot be more than T terms. Thus, we have proved (b). ■

Our proof is naturally general: that of (a) would be no shorter even if restricted to $r = 1$ (but the deduction of (b) would become trivial). What is more, it is applicable to projective geometries [1]. Furthermore, our proof, even restricted to $r = 1$, is simpler than the original proofs by Meshalkin and Hochberg–Hirsch.

The upper bounds in the theorem can be attained only in limited circumstances. When $r = 1$, the maxima are attained if for each k , $\mathcal{M}_k = \mathcal{P}_{\lfloor n/p \rfloor}(S)$ or $\mathcal{P}_{\lceil n/p \rceil}(S)$ [6]. When $p = 2$, the upper bounds are attained if \mathcal{M}_1 is the union of the r largest classes $\mathcal{P}_m(S)$ [3]. When $r > 1$ and $p > 2$, the upper bounds are only sometimes attainable, but proving this is complicated.

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