Hao's Problem: Solution

There are three possible outcomes: (N) in every matching J is in no box, (Y) in every matching J is in some box, and (?) in some matchings J is and in others J is not in a box. We want to distinguish among them based on the forbidden lists.

We have a set $\mathcal{B}$ of 9 boxes $B_1, \ldots, B_9$ and a set $\mathcal{T}$ of 10 toys, $A, \ldots, J$. Define the relation $R : \mathcal{B} \to \mathcal{T}$ by $R = \{(B_i, T) : \text{toy } T \text{ may be in } B_i\}$ (using the lists). We know a matching exists since there are 9 toys in the 9 boxes, so R satisfies Hall's condition. That is, $|R(K)| \geq |K|$ for every $K \subseteq \mathcal{B}$.

What prevents a matching from matching to J? Suppose $K \subseteq \mathcal{B}$ and $J \notin R(K)$; then no box in K can contain J. In particular, if $J \notin R(\mathcal{B})$, then J is not in a box. That is equivalent to saying J is in every forbidden list.

On the other hand, if for some critical set K we have $J \in R(K)$, then J must be in a box in K. (Recall that K is critical if $|R(K)| = |K|$; then K must match onto $R(K)$ in any matching.) Since the union of critical sets is critical, there is a largest critical set L that contains all others; and if $J \in R(L)$, then J must be in a box. This is equivalent to seeing whether after deleting J we still satisfy Hall's condition; if not, J must be in a box. In other words, we only need to find the largest critical set L and find whether J is in $R(L)$.

If $J \notin R(L)$, then J is not in a box of L. That doesn't answer the question unless $L = \mathcal{B}$.

Otherwise, I think we can't tell whether or not J is in a box. Specifically, if $L \subset \mathcal{B}$ and $J \notin R(L)$, we can't tell. That needs to be verified.