DUALITY
MATH 580, FALL 2001

Version of September 25, 2001.

1. Abstract Duality

I propose this problem as a group research project:

(1) Figure out a truly self-dual axiom system based on circuits and cocircuits and the property
$$|C \cap C^*| \neq 1 \quad \forall C \in \mathcal{C},\ C^* \in \mathcal{C}^*.$$

2. Duality of Vector Representations

Here is a list of problems that will help you to understand the details of the proofs I gave in class. (I didn't provide all the details, and the ones I did give weren't necessarily presented in the best way. Here, e.g., I am trying to make parts of the proof into lemmas (presented as exercises) that may be useful in other situations than the particular proofs they were contained in in class.)

The theorems of concern here are:

- Theorem 2.2.8.
- *Theorem 2.2.A. Let the scalar field be the real numbers. Given an $r \times n$ matrix A whose column matroid $M[A]$ has rank r. Let S be an $n \times n$ matrix whose first r rows are A, and let A^* be the matrix composed of the last $n-r$ rows of S. If S is nonsingular and every row of A^* is orthogonal to every row of A, then $M[A^*] = M^*[A]$.*
- *Theorem 2.2.B. In the Euclidean vector space $\mathbb{R}^n$ (with dot product), let $b_1, \ldots, b_n$ be an orthonormal basis and let W be a subspace. Let y_i be the orthogonal projection of b_i onto W and let z_i be its orthogonal projection onto $W^\perp$. Let M be the vector matroid of $y_1, \ldots, y_n$. Then the vector matroid of $z_1, \ldots, z_n$ is M^*.*

The following problems formulate portions of the proofs of these theorems (Theorem 2.2.A only as of September 25, 2001).

(1) Let A be any $m \times n$ matrix. Take a subset of the rows of A to form a new matrix A'. Prove: *$M[A'] = M[A]$ if and only if the rows of A' span the row space of A (i.e., $\mathcal{R}(A') = \mathcal{R}(A)$).* (Valid over any field.)

(2) Let A be any $m \times n$ matrix. Let E be an $m \times m$ matrix. Prove: *If E is invertible, then $M[EA] = M[A]$.* (Valid over any field.)

(3) Suppose A is as in the statement of Theorem 2.2.A. Prove that an S exists, i.e., that it is always possible to find A^*. (I suggest you do this by looking up theorems in linear algebra, not by proving the linear algebra yourself.)

(4) Fill in the details of the matrix theory of the last part of the proof of Theorem 2.2.A. This is the part where I proved that, assuming the first r columns of A make a nonsingular matrix, then the last $n-r$ columns of A^* make a nonsingular matrix. (My details in class were not complete and not all quite correct.)

(5) (a) Using Problem 4, complete the proof that $\mathcal{B}(m[A^*]) \supseteq \mathcal{B}'(M[A])$.
(b) Complete the proof that $\mathcal{B}(m[A^*]) = \mathcal{B}'(M[A])$.
(c) Complete the proof that $\mathcal{B}(m[A^*]) = \mathcal{B}(M^*[A])$.

(6) With I and D as in Theorem 2.2.8, prove that every row of $[\ I \mid D\ ]$ is orthogonal (in the dot product) to every row of $[\ -D^{\mathrm{T}} \mid I\ ]$. More strongly: prove that

$$\mathcal{R}[\ -D^{\mathrm{T}} \mid I\ ] = (\mathcal{R}[\ I \mid D\ ])^{\perp}.$$

(7) Using Problem 6, show that Theorem 2.2.8 is a special case of Theorem 2.2.A. (This is related to Proposition 2.2.23.)

(8) A theorem of linear algebra states: Suppose you have a subspace W of $\mathbb{R}^n$ that is generated by a basis $\alpha_1, \ldots, \alpha_r$ and you form the matrix A whose rows are the vectors α_i, $i = 1, \ldots, r$. Then the orthogonal projection onto W of any vector $x \in \mathbb{R}^n$ is given by the formula $\operatorname{proj}_W x = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}Ax$. Find out how to prove this formula, either by looking it up or by working it out yourself.

(9) Show that, in the proof of Theorem 2.2.B given in class, $\mathcal{R}(A^*) = \mathcal{R}(A)^{\perp}$.

(10) In the proof of Theorem 2.2.B, show that $M[Z] = M[A^*]$. (This is a step I omitted in class.)