HOMEWORK ON SIGNED, GAIN, AND BIASED GRAPHS

1. Signed graphs

1. Write a complete proof of **Harary's Bipartition Theorem ("Structure Theorem")** for signed graphs: Σ is balanced $\iff$ it has no halfedges and there is a bipartition $V = V_1 \mathbin{\dot\cup} V_2$ such that every positive edge has both endpoints in the same part (V_1 or V_2) and every negative edge has one endpoint in each part. You may use the outline given in class, or find a different proof.
2. Prove the **Lemma** mentioned in class: Every circle in Σ is positive $\iff$ every circle of a fundamental system is positive.

   (Recall that, given a spanning tree T of a connected graph and an edge $e \notin T$, the *fundamental circle* of e with respect to T is $C_T(e) :=$ the unique circle in $T \cup e$. The *fundamental system of circles* with respect to T is the set of all circles $C_T(e)$. A *fundamental system of circles* of a graph is any fundamental system with respect to any spanning tree.)
3. Prove the **Proposition** (a sort of corollary of the bipartition theorem, also due to Harary) that, if Σ has no halfedges, then Σ is balanced $\iff$ for every pair of vertices v, w, the sign of a path joining v and w is independent of the path (it depends only on the pair v, w).
4. Suppose Φ is a gain graph with gain group equal to F^* for some field F. Give a complete proof that the vectors correspond to a contrablanced theta subgraph are a minimal dependent set of vectors.
5. Define switching in a gain graph Φ over an arbitrary group $\mathfrak{G}$. Then use switching to prove that an edge set is balanced if and only if it has no half edges and it switches so that all gains equal 1, the group identity. (Part of the problem is to figure out how to define switching. However, if you don't like that part, you can look it up in "BG1". The proof is also in "BG1"; if you read it there, you probably shouldn't submit it as homework.)
6. Take a gain graph Φ with finite gain group $\mathfrak{G}$. Prove that the improper edge set $I(\kappa)$ of a k-coloring is closed in $G(\Phi)$ and if κ is zero-free then $I(\kappa)$ is also balanced; and conversely.
7. Prove: **Theorem.** For a gain graph Φ with finite gain group $\mathfrak{G}$, we have

$$\chi_\Phi(\lambda) = \sum_{U \in \operatorname{Lat}\Phi} \mu(\varnothing, U)\lambda^{b(U)},$$

$$\chi^*_\Phi(\lambda) = \sum_{U \in \operatorname{Lat}^{\mathrm{b}}\Phi} \mu(\varnothing, U)\lambda^{b(U)}.$$

   Hint: Use improper edge sets of colorings, and Möbius inversion. You may use the previous problem.

8. We have a simple graph Δ and a group $\mathfrak{G}$ (all these are finite), and we form the group expansion $\mathfrak{G} \cdot \Delta$ and the corresponding biased graph $\Omega = \langle \mathfrak{G} \cdot \Delta \rangle$. Find a simple formula for the zero-free chromatic polynomial $\chi^*_\Omega(\lambda)$ in terms of Δ.

9. Let Φ be the gain graph corresponding to the extended Catalan arrangement,
$$\mathcal{A}_k = \{x_j - x_i = c : i \neq j \text{ and } c = 0, \pm 1, \ldots, \pm k\},$$
where k is a positive integer and we're in n-dimensional space. First, write down what Φ is, explicitly. Then, calculate $\chi_\Phi(\lambda)$ by the method of shrinking the gain group, as explained in class.

10. Write down a complete proof that, if Φ is a gain graph with no halfedges or unbalanced loops and if the gain group is finite, then $\chi_{\Phi^\bullet}(\lambda) = \chi^*_\Phi(\lambda - 1)$. If ambitious, generalize to all finite gain graphs or more generally all finite biased graphs.