EHRHART HOMEWORK

1. Chapter 3

1. Show that a cone K is pointed if and only if there is a hyperplane H such that $H \cap K$ is a single point. (Note that H is a supporting hyperplane of K, by definition.) Furthermore, if so, then that point is the apex v.

2. Show that if K contains an affine subspace A, then it contains an affine subspace A' parallel to A (this means: a translate of A) such that there is a supporting hyperplane H with $H \cap K \supseteq A'$. (This leads to a generalization of the concept of apex to nonpointed cones, which you may think about if you like.)

3. Suppose H is a supporting hyperplane of K such that $H \cap K = \{v\}$, and that $v+w \in K$. Show that $P := (H+w) \cap K$ is a convex polytope. Show that $K_v(P) = K$.

4. Show that if P is a convex polytope, x is in the relative interior of a face F of P. and x is visible from a point $v \notin P$, then every point of F is visible from v.

5. Show that, if P is a convex polytope and $x \in P$ is visible from a point $v \notin P$, then some facet of P containing x is also visible from v.

2. Chapter 4

1. Strengthen Exercise 3.13 as follows (which seems to be needed for a complete proof of Theorem 4.3): Let K be a rational d-cone with apex 0 and let $K_1, \dots, K_m$ be the cones of a triangulation of K. Show that there exists a vector $v \in \mathbb{R}^d$ (which can moreover be chosen arbitrarily small) such that

$$(v + K) \cap \mathbb{Z}^d = K^\circ \cap \mathbb{Z}^d, \tag{4.4}$$

$$\partial(v + K_j) \cap \mathbb{Z}^d = \varnothing \text{ for all } j = 1, \dots, m, \tag{4.5}$$

$$\partial(-v + K_j) \cap \mathbb{Z}^d = \varnothing \text{ for all } j = 1, \dots, m. \tag{4.5$'$}$$

2. Find the polar duals P^* of the planar convex polytopes (I mean, convex polygons):
   a. $P = \text{conv}\{\pm e_1, \pm e_2\}$ in $\mathbb{R}^2$.
   b. $P = \{x \in \mathbb{R}^2 : |x_1| \le 1, |x_2| \le 1\}$.
   Can you generalize these to higher dimensions?

3. Consider a convex d-polytope $P \subset \mathbb{R}^d$ with $0 \in P^\circ$. Show that P^* is integral if and only if P has the form $\{x \in \mathbb{R}^d : Ax \le \mathbf{1}\}$ for some integral matrix A.

3. Chapter 5

1. To which rational polytopes can you generalize Theorem 5.4? (Hint: Try half-integral, third-integral, quarter-integral, etc.)

4. Chapter 6

1. Explain the period of $M_3(t)$ by finding the vertices of the polytope of 3×3 magic squares.