HOMEWORK ON ARRANGEMENTS OF HYPERPLANES AND INSIDE-OUT POLYTOPES

1. Prove Lemma 1 by induction on $r(y)$.
2. Prove Lemma 2. (Relatively hard.)
3. Prove **Lemma 3**: $f_d(\mathcal{H})$, the number of regions of $\mathcal{H}$, obeys the law
$$f_d(\mathcal{H}) = f_d(\mathcal{H} \setminus \{h\}) + f_{d-1}(\mathcal{H}^h),$$
where $\mathcal{H}^h$ denotes the arrangement induced in h by $\mathcal{H}$.
4. Prove Theorem 2 using Lemma 2. (Hint: Use Lemma 3.)
5. Prove the chromatic polynomial of a graph obeys the law
$$\chi_\Gamma(\lambda) = \chi_{\Gamma\setminus e}(\lambda) - \chi_{\Gamma/e}(\lambda)$$
for every edge e.
6. Prove that, for a link e in Γ, $\mathcal{H}[\Gamma/e] = \mathcal{H}[\Gamma]^{h_e}$, where h_e is the hyperplane that corresponds to edge e.
7. Prove that a tree T of order d has chromatic polynomial $\lambda(\lambda-1)^{d-1}$.
8. Prove that, if e is an isthmus in a graph Γ, then
$$\chi_\Gamma(\lambda) = (\lambda - 1)\chi_{\Gamma\setminus e}(\lambda).$$
(This gives a generalization of the previous question.)
9. Prove that, if $h \in \mathcal{H}$ and $r(\mathcal{H} \setminus \{h\}) \neq r(\mathcal{H})$, then $r(\mathcal{H} \setminus \{h\}) = r(\mathcal{H}) - 1$ and
$$p_{\mathcal{H}}(\lambda) = (\lambda - 1)p_{\mathcal{H}\setminus\{h\}}(\lambda).$$
10. Use the previous problems to assemble a complete proof of Theorem 3.
11. Prove **Lemma 8**: In a poset A, define $\mu^*(x, y)$ by
$$\mu^*(x,y) = \begin{cases} 0 & \text{if } x \not\leq y, \\ 1 & \text{if } x = y, \\ -\sum_{z:x<y\leq z} \mu^*(z,y) & \text{if } x < y. \end{cases}$$
Then $\mu^* = \mu$.
12. Show that a hyperplane is rational, in the sense of having an equation with integral coefficients, if and only if it is affinely generated by rational points.
13. Prove that, for any affine subspace $u \subseteq \mathbb{R}^d$ and convex polytope P, we have
$$\operatorname{aff}(u \cap P^\circ) = u \cap \operatorname{aff}(P).$$

14. Open question: Is it true that $\mathcal{L}(P^\circ, \mathcal{H})$, defined as
$$\{P^\circ \cap \left(\bigcap \mathcal{S}\right) : P^\circ \cap \left(\bigcap \mathcal{S}\right) \neq \varnothing\},$$
is always isomorphic as a poset to
$$\{\left(\bigcap \mathcal{S}\right) : P^\circ \cap \left(\bigcap \mathcal{S}\right) \neq \varnothing\}?$$

15. Use the geometrical argument outlined in class, along with Rota's sign theorem for the Möbius function, to show that the chromatic polynomial $\chi_\Gamma(k) = \sum_{d=c}^{n} a_d k^d$ where c = number of components of the graph Γ, n = number of vertices, and $(-1)^{n-d} a_d > 0$ (sign property).

16. Let $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$ and define $h_i^+ := \{y \in \mathbb{R}^n : x_i \cdot y > 0\}$. Prove: **Theorem.** The intersection $h_1^+ \cap h_2^+ \cap \cdots \cap h_n^+ = \varnothing \iff$ there is a positive dependence among $x_1, x_2, \ldots, x_n$.