The Acyclotope

by Thomas Zaslavsky

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Let $\Gamma = (V, E)$ be a graph. We'll allow loops and multiple edges until further notice, but all graphs are finite. We write $V = \{v_1, \ldots, v_n\}$. An edge with endpoints v_i and v_j may be written $v_i v_j$ or e_{ij}; if there are multiple edges this notation will not determine the edge, but it is still useful to quickly show the endpoints. A connected graph with degree 2 at every vertex is called a *circle* (not "cycle" as is all too common in graph theory; we have other uses for the name "cycle").

Definition 1. *[{D:acyc}]* The *acyclotope* $A(\Gamma)$ is the convex hull of net degree vectors of acyclic orientations of Γ.

We have to define the terms in the definition.

An orientation of Γ is an assignment of a direction to each edge. We indicate the direction in two ways: by arrows at the ends of the edge e_{ij}, one arrow pointing away from the vertex and the other into the vertex. The oriented graph is notated $\vec{\Gamma}$ and an oriented edge is written $\vec{e}_{ij}$, meaning that v_i is the tail and v_j is the head. We assume the two ends of an edge are distinguishable, even for a loop, so every edge has two orientations.

I want to distinguish between an *oriented graph*, where the orientation is variable, and a directed graph ("digraph"), where the orientation is a given, not to be changed. The questions are different. However, in many ways there is no difference; e.g., indegrees, et al.

The *degree vector* of Γ is $d(\Gamma) = (d_1(\Gamma), \ldots, d_n(\Gamma))$ where $d_i = d(v_i) = d_i(\Gamma)$ is the number of edges incident with v_i, but a loop counts twice. (I should have said it's the number of *edge ends* incident with v_i, but that's unconventional. Assume I meant that.) The *indegree vector* of an oriented graph $\vec{\Gamma}$ is $d^+(\vec{\Gamma})$ whose components are $d_i^+ = d_i^+(\vec{\Gamma}) = d^+(v_i) =$ the number of incoming edge ends at v_i. The *outdegree vector* is $d^-(\vec{\Gamma})$, defined similarly. The *net degree vector* is $d^\pm(\vec{\Gamma}) = d^+(\vec{\Gamma}) - d^-(\vec{\Gamma})$.

1. The Permutahedron

The *permutahedron* of degree n is a famous polytope. It is the convex hull of all permutations of $[n]$, i.e.,

$$\Pi_{n-1} := \operatorname{conv}\{(1^\pi, 2^\pi, \ldots, n^\pi) : \pi \in \mathfrak{S}_n\}.$$

Here $\mathfrak{S}_n$ is the symmetric group of degree n (i.e., on n symbols, specifically on $[n]$).

Digression on permutations. The classical viewpoint is that a permutation is a rearrangement of a linearly ordered set; this becomes the more abstract algebraic viewpoint that a permutation is a self-bijection. A combinatorial view is that a permutation is a linear ordering. Let's stay with the set $[n]$, which is naturally ordered. Then the two viewpoints converge. A bijection $\pi : [n] \to [n]$ can be written as the sequence $(1^\pi, 2^\pi, \ldots, n^\pi)$, known as the *one-line representation* of π, and every such sequence comes from a unique bijection. (I like to think of the connecting link as the two-line representation $\begin{pmatrix} 1 & 2 & \cdots & n \\ 1^\pi & 2^\pi & \cdots & n^\pi \end{pmatrix}$, which explicitly represents any algebraic permutation and which, without the first line, is a combinatorial permutation. I write Perm_n for the set of permutation sequences and $\mathfrak{S}_n$ for the group of bijections. Thus, $\Pi_{n-1} := \mathrm{conv}\,\mathrm{Perm}_n$.

I like to modify the permutahedron in two ways. Define

$$\Pi'_{n-1} := \Pi_{n-1} - \mathbf{1} = \mathrm{conv}\{(0^\pi, 1^\pi, \ldots, [n-1]^\pi) : \pi \in \mathfrak{S}_n\}$$

and

$$\Pi''_{n-1} := 2\Pi'_{n-1} + (n-1)\mathbf{1} = \mathrm{conv}\{([-n+1]^\pi, [-n+3]^\pi, \ldots, [n-1]^\pi) : \pi \in \mathfrak{S}_n\}.$$

Here $\mathbf{1}$ denotes the all-1's vector. The standard orthonormal basis vectors of $\mathbb{R}^n$ are $b_1, \ldots, b_n$.

Theorem 1. [{T:perm}] *The permutahedron is an $n-1$-dimensional zonotope with vertex set* Perm_n. *A zonotopal representation is given by*

$$\Pi''_{n-1} = \sum_{i<j\in[n]} [b_i - b_j, b_j - b_i].$$

This is a special case of a general theorem about the acyclotope of a graph. First, let's look at the meaning of the transformations of Perm_n to $\mathrm{Perm}'_n := \{(0^\pi, 1^\pi, \ldots, [n-1]^\pi) : \pi \in \mathfrak{S}_n\}$ and to $\mathrm{Perm}''_n := \{([-n+1]^\pi, [-n+3]^\pi, \ldots, [n-1]^\pi) : \pi \in \mathfrak{S}_n\}$.

Proposition 2. [{P:movedPerm}] $\mathrm{Perm}'_n = \{d^-(\vec{K}_n) : \vec{K}_n$ *is an acyclic orientation of* $K_n\}$, *and* $\mathrm{Perm}''_n = \{d^\pm(\vec{K}_n) : \vec{K}_n$ *is an acyclic orientation of* $K_n\}$.

Side note: A directed complete graph is known in graph theory as a *tournament*.

Proof. Theorem 6(a) implies that the acyclic orientations of K_n are derived from linearly ordering V. From that it is clear that the outdegree vector is a permutation of $\{0, 1, \ldots, n-1\}$ and the net degree vector is a permutation of $\{-(n-1), -(n-3), \ldots, n-3, n-1\}$. ☐

Proof of Theorem 1. By Proposition 2, $\Pi''_{n-1} = A(K_n)$. Thus the theorem is a special case of Theorem 7. ☐

2. Acyclic and Cyclic Orientations

A *cycle* $\vec{C}$ is a circle C oriented so that every edge goes the same way around C. If $C = e_{01}e_{12}\cdots e_{l-1,l}$ where $v_0 = v_l$ (and $l > 0$; a circle cannot have length 0), the cycle is $\vec{e}_{01}\vec{e}_{12}\cdots\vec{e}_{l-1,l}$ or its reverse, $\vec{e}_{l,l-1}\cdots\vec{e}_{21}\vec{e}_{10}$. Note that each vertex has indegree 1 and outdegree 1, so it has no *source* (a vertex with no outgoing edges) or *sink* (a vertex with no incoming edges). (An isolated vertex is a source and a sink.)

An oriented graph is *acyclic* if it has no cycles. Otherwise it is (surprise!) *cyclic*. (It is *totally cyclic* if every edge belongs to a cycle. Total cyclicity is dual to acyclicity, but I won't go into that.) An acyclically oriented graph cannot have loops, because an oriented loop is already a cycle.

Suppose we partially order the vertex set V by $\prec$, which is *suitable for* Γ in the sense that for every edge vw the vertices v and w are comparable. Then we orient Γ by choosing the orientation $\vec{e}_{ij}$ if $v_i \prec v_j$. We call this the orientation *implied by* $\prec$. (Suitability is just the requirement that ensures every edge is oriented. Note that we could assume $\prec$ is a total ordering, since every partial ordering extends to a total ordering.)

Lemma 3. [{L:po}] *The orientation implied by a suitable partial ordering of V is acyclic.*

Proof. Suppose there were a cycle $e_{01}e_{12}\cdots e_{l-1,l}$, where $v_0 = v_l$. Then $v_0 \prec v_1 \prec \cdots \prec v_l$, but by transitivity then $v_0 \prec v_l = v_0$, which violates reflexivity. ☐

There is a kind of converse.

Lemma 4. [{L:acyc}] *Suppose $\vec{\Gamma}$ is acyclic. Define $v_i \prec_{\vec{\Gamma}} v_j$ if there is a (directed) path from v_i to v_j. Then $\prec_{\vec{\Gamma}}$ is a partial ordering of V; for every edge there is one orientation $\vec{e}_{ij}$ such that $v_i \prec_{\vec{\Gamma}} v_j$; and that is the orientation of e_{ij} in $\vec{\Gamma}$.*

Furthermore, $\prec_{\vec{\Gamma}}$ is the unique minimal partial ordering that implies the orientation $\vec{\Gamma}$ of Γ; that is, a partial ordering that implies $\vec{\Gamma}$ contains $\prec_{\vec{\Gamma}}$.

Proof. NEEDS PROOF

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Lemma 5. [{L:sinksource}] *An acyclically oriented graph has a source and a sink.*

Proof. Take the minimal associated partial ordering $\prec_{\vec{\Gamma}}$. The sources are the minimal elements of $\prec_{\vec{\Gamma}}$ and the sinks are the maximal elements of $\prec_{\vec{\Gamma}}$. ☐

The Hasse diagram of a partially ordered set $(V, <)$ is a directed graph if we orient all edges upwards. The *comparability graph* of $(V, <)$ is the graph on V with an edge vw if and only if v and w are comparable; by directing every edge upwards, that is, $\overrightarrow{vw}$ if $v < w$, we get the *comparability digraph* Comp($<$).

Theorem 6. [{T:ao}] *Consider the orientations of a graph Γ.*

(a) *An orientation is acyclic if and only if it is derived from a partial ordering of V.*
(b) *An orientation $\vec{\Gamma}$ is acyclic if and only if it satisfies* $\mathrm{Hasse}(\prec) \subseteq \vec{\Gamma} \subseteq \mathrm{Comp}(\prec)$ *for some partial ordering $\prec$ of V.*
(c) *Suppose $\vec{\Gamma}$ is an orientation of Γ. If $\vec{\Gamma}$ is acyclic, there is no other orientation with the same net degree vector. If $\vec{\Gamma}$ is acyclic, there does exist another orientation with the same net degree vector.*

Proof of (a). The forward direction is Lemma 4. The backward direction is Lemma 3. ☐

Proof of (b). If $\vec{\Gamma}$ is acyclic, then $\prec_{\vec{\Gamma}}$ is a partial ordering $\prec$ with the property stated in the theorem.

Conversely, if there exists such a partial ordering $\prec$, then MORE MORE MORE

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Proof of (c). First, suppose $\vec{\Gamma}$ is acyclic. (We don't exclude the possibility that $\vec{\Gamma}' = \vec{\Gamma}$.) Then $\vec{\Gamma}$ has a source. The indegree of a source v_i is 0, so the net degree is $-d(v_i)$; conversely, if $d^{\pm}(v_i) = -d(v_i)$, then v_i is a source. Thus, we can identify a source from $d^{\pm}(\vec{\Gamma})$.

Now define $\vec{\Gamma}' := \vec{\Gamma} \setminus v_i$. We can predict the net degree vector of $\vec{\Gamma}'$ from that of $\vec{\Gamma}$ and the knowledge that v_i is a source; specifically,

$$d_j^{\pm}(\vec{\Gamma}') = \begin{cases} d_j^{\pm}(\vec{\Gamma}) & \text{if there is no edge } v_iv_j, \\ d_j^{\pm}(\vec{\Gamma}) - 1 & \text{if there is an edge } v_iv_j, \end{cases}$$

since any edge v_iv_j is oriented towards v_j. It follows that, if no other orientation of $\Gamma' = \Gamma \setminus v_i$ has the same net degree vector as $\vec{\Gamma}'$, then the orientation $\vec{\Gamma}$ is determined. Hence, by induction on n, $\vec{\Gamma}$ is determined by its net degree vector.

For the second part, suppose $\vec{\Gamma}$ has a cycle $\vec{e}_{01}\vec{e}_{12}\cdots\vec{e}_{l-1,l}$. Define $\vec{\Gamma}'$ to be $\vec{\Gamma}$ with the cycle edges reversed. That does not change the net degree vector; hence $d^{\pm}(\vec{\Gamma}) = d^{\pm}(vG')$. $\square$

The proof shows that we can identify all sources (and similarly all sinks) of an orientation from the net degree vector; by repeatedly stripping them out, we can tell algorithmically from $d^{\pm}(\vec{\Gamma})$ whether $\vec{\Gamma}$ is cyclic or acyclic. I don't know whether that can be done more directly.

3. The Acyclotope is a Zonotope

A convex polytope is the convex closure of a finite set of points: $P = \text{conv}(S)$ where $S \subset \mathbb{R}^n$ is finite. A vertex is a point $p \in S$ that is necessary to get the right convex hull; i.e., $\text{conv}(S \setminus p) \subset P$. (This is not the definition but it's a property proved in courses on convex polytopes.) The vertex set of P is denoted by $\text{Vert}(P)$.

We assign to an edge e_{ij} the vector $x(e_{ij}) := b_j - b_i$ in $\mathbb{R}^n$, the vertex space, where (as before) $b_1, \ldots, b_n$ is the standard orthonormal basis. Since we didn't orient the edge, this definition allows either $b_j - b_i$ or $b_i - b_j$; that won't matter. If it does matter, we orient the edge and define $x(\vec{e}_{ij}) := b_j - b_i$. (Note that a loop gets the zero vector.)

Theorem 7. [{T:acyclotope}] *The acyclotope $A(\Gamma)$ has the following properties.*
(a) $A = \sum_{e_{ij} \in E; i<j} [-x(e_{ij}), +x(e_{ij})]$.
(b) $A = \text{conv}\{d^{\pm}(\vec{\Gamma}) : \vec{\Gamma}$ *is an orientation of* $\Gamma\}$.
(c) $\text{Vert}(A) = \{d^{\pm}(\vec{\Gamma}) : \vec{\Gamma}$ *is an acyclic orientation of* $\Gamma\}$.

Part (i) shows that the acyclotope is a zonotope.

Proof. We begin with a definition. Let

$$A' := \text{conv}\{d^{\pm}(\vec{\Gamma}) : \text{all orientations } \vec{\Gamma}\}.$$

We'll prove some properties of A' and that $A = A'$.

Lemma 3.1. [{L:A'zono}] $A' := \sum_{e_{ij}} [-x(e_{ij}), x(e_{ij})]$.

Proof. $\square$

Lemma 3.2. [{L:A'zono}] *If $\vec{\Gamma}$ is cyclic, then $d^{\pm}(\vec{\Gamma})$ is not a vertex of A'.*

Proof. $\square$

Now, two lemmas about A.

Lemma 3.3. [{L:acyclicvert}] *If $\vec{\Gamma}$ is acyclic, then $d^{\pm}(\vec{\Gamma})$ is a vertex of $A(\Gamma)$.*

Proof. Since $\vec{\Gamma}$ is acyclic, it has a source. The indegree of a source v_i is 0, so the net degree is $-d(v_i)$; conversely, if $d^{\pm}(v_i) = -d(v_i)$, then v_i is a source. Thus, we can identify a source from $d^{\pm}(\vec{\Gamma})$.

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Lemma 3.4. [{L:cyclicnovert}] *If $\vec{\Gamma}$ is cyclic, then $d^{\pm}(\vec{\Gamma})$ is not a vertex of $A(\Gamma)$.*

Proof. Suppose $\vec{\Gamma}$ has a cycle $\vec{e}_{01}\vec{e}_{12}\cdots\vec{e}_{l-1,l}$. Define $\vec{\Gamma}'$ to be $\vec{\Gamma}$ with reoriented edge $\vec{e}_{10}$ instead of $\vec{e}_{01}$. Then

$$d_i^{\pm}(\vec{\Gamma}') = \begin{cases} d_i^{\pm}(\vec{\Gamma}) & \text{if } v_i \neq v_0, v_1, \\ d_i^{\pm}(\vec{\Gamma}) + 2 & \text{if } v_i = v_0, \\ d_i^{\pm}(\vec{\Gamma}) - 2 & \text{if } v_i = v_1. \end{cases}$$

Define $\vec{\Gamma}''$ to be $\vec{\Gamma}$ with reoriented edges $\vec{e}_{21}, \ldots, \vec{e}_{l,l-1}$; then

$$d_i^{\pm}(\vec{\Gamma}'') = \begin{cases} d_i^{\pm}(\vec{\Gamma}) & \text{if } v_i \neq v_0, v_1, \\ d_i^{\pm}(\vec{\Gamma}) - 2 & \text{if } v_i = v_0, \\ d_i^{\pm}(\vec{\Gamma}) + 2 & \text{if } v_i = v_1. \end{cases}$$

Since $d^{\pm}(\vec{\Gamma}) = d^{\pm}(\vec{\Gamma}') + d^{\pm}(\vec{\Gamma}'')$, the net degree vector of $\vec{\Gamma}$ is not a vertex of $A(\Gamma)$. □

The proof of Theorem 7 continues ...
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References

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