MATH 580A: CONVEX POLYTOPES
ADDITIONAL EXERCISES

I. Show that the permutation vector $(1, 2, ..., d)$ is the intersection of the permutahedron Π_{d-1} with the hyperplane $x_1 + \frac{1}{2}x_2 + \cdots + \frac{1}{d}x_d = d$. (Note that Π_{d-1} lies in the positive ($\geq$) half-space of this hyperplane.)

Deduce that $\mathrm{Vert}(\Pi_{d-1})$ is the set of permutation vectors. (This part is optional, since I don't recall specifically assigning it, but it's good if you did it.)

II. Why did I use the net degree vector to define the acyclotope $A(\Gamma)$, and not the in-degree vector as we did for the permutahedron (strictly speaking, for $\Pi_{d-1} - \mathbf{1}$)?

III. Prove the formula on page 4 for $\mathrm{conv}(K)$ when K is finite.

IV. P is a convex polyhedron. Prove that

$$\begin{aligned}\mathrm{rec}(P) &= \bigcup \big\{ \mathrm{pos}(\mathbf{y}) : (\exists \mathbf{x} \in P)\ \mathbf{x} + \mathrm{pos}(\mathbf{y}) \subseteq P \big\} \\ &= \big\{ \mathbf{y} : (\exists \mathbf{x} \in P)\ (\forall t \in \mathbb{R}_{\geq 0})\ \mathbf{x} + t\mathbf{y} \in P \big\}.\end{aligned} \tag{1}$$

V. P is a convex polyhedron. Prove that

$$\begin{aligned}\mathrm{lineal}(P) &= \bigcup \big\{ \mathrm{lin}(\mathbf{y}) : (\exists \mathbf{x} \in P)\ \mathbf{x} + \mathrm{lin}(\mathbf{y}) \subseteq P \big\} \\ &= \big\{ \mathbf{y} : (\exists \mathbf{x} \in P)\ (\forall t \in \mathbb{R})\ \mathbf{x} + t\mathbf{y} \in P \big\}.\end{aligned} \tag{2}$$

VI. Find a good definition of $\mathrm{lineal}(\varnothing)$. Explain why you think it's good.

VII. Ziegler makes many unproved assertions, which he assumes are elementary. You, as a reader, ought to look for such assertions and see if you can prove them. Here are the ones I noticed about the lineality space. P is any polyhedron, H- or V-, possibly empty.

(a) **Lemma A.** $\mathrm{lineal}(P)$ *is a homogeneous subspace of* $\mathbb{R}^d$.

(b) **Lemma B.** *Let* U *be any linear subspace complementary to* $\mathrm{lineal}(P)$. *Then* $\mathrm{lineal}(P \cap U) = \{\mathbf{0}\}$.

(c) **Proposition C.** $P = \mathrm{lineal}(P) + (P \cap U)$. *Moreover, the representation* $\mathbf{x} = \mathbf{y} + \mathbf{z}$ *for* $\mathbf{x} \in P$, *where* $\mathbf{y} \in \mathrm{lineal}(P)$ *and* $\mathbf{z} \in P \cap U$, *is unique.*

(d) **Corollary D.** $P \cap U = \varnothing$ *iff* $P = \varnothing$.

(e) **Proposition E.** *For an H-polyhedron* $P = P(A, \mathbf{z})$, $\mathrm{lineal}(P) = \mathrm{Nul}\, A$. ($\mathrm{Nul}\, A := \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$ is the null space of A.)

Prove Proposition C and any one other of these except Lemma A (which we did in class).

VIII. Find the polar dual, $P^{\triangle}$.

(a) P_1 is the intersection of half-spaces $x \geq -2$, $y \geq -1$, $x - y \leq 2$, and $2y - x \leq 10$.

(b) $P_2 = \text{conv}\{(1,3), (-1,3), (-2,-3), (-1,-3)\}$.

(c) P_3 is P_2 translated by $(6,6)$. Compare $P_3^{\triangle}$ with $P_2^{\triangle}$. Can you identify an effect of translation?

IX. Prove that any dependency λ is a linear combination of minimal dependencies (circuit dependencies), such that the circuits are all contained in $\operatorname{supp}\lambda$. Your proof should apply to both linear and affine dependencies.

(This is a warm-up problem for what we want for Ch. 6, where we want the circuit dependencies to conform to λ.)

X. Let

$$A = \begin{pmatrix} -I_5 \\ B \end{pmatrix}$$

where

$$B = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

and let

$$\mathbf{z} = \begin{pmatrix} \mathbf{0}_5 \\ \mathbf{1}_4 \end{pmatrix}.$$

Find all the vertices of $P(A, \mathbf{z})$. (You should check that it's bounded.) Are they integral?