Proofs About the Orlik–Solomon Algebra A of a Matroid

Throughout, $\mathcal{M}$ is a simple matroid. The ground set $\mathcal{E}$ is linearly ordered. A broken circuit is a set $C \setminus \max C$, where C is a circuit and $\max C$ denotes the largest element of C in the ground-set linear ordering. Recall that we defined $\mathrm{NBC}(\mathcal{M})$ to be the class of no-broken-circuit sets (NBC-sets, not sometimes called "peacock sets"), i.e., sets $S \subseteq \mathcal{E}$ that do not contain any broken circuit. We defined $\mathrm{NBC}_p = \{S \in \mathrm{NBC} : |S| = p\}$.

In the algebras, when I write for instance a_S where S is a sequence of matroid elements, I will assume they are in any order, i.e., I don't distinguish between different orderings of S. The specific ordering can't affect anything but the sign, so all orderings give the same basis element for practical purposes. I can make this formal: assume the chosen S is the one in the ground-set linear order.

Theorem 0.1. *The elements a_N for $N \in \mathrm{NBC}_p(\mathcal{M})$ form a basis for A_p.*

Proof. Consider a circuit $C = \{h, h_1, h_2, \ldots, h_k$ where $h = \max C$ (by choice of notation). Then $\partial e_C = e_{C\setminus h} + \sum_i \pm e_{C\setminus h_i}$, and $\partial e_C \in I$, so $a_{C\setminus h} = \sum_i \pm a_{C\setminus h_i}$. $C \setminus h$ is a broken circuit; this shows that to generate A as a vector space we don't need any broken circuit terms. Since I is an ideal, we could have said we don't need any terms $a_S a_{C\setminus h}$, since $a_S a_{C\setminus h} = sum_i \pm a_S a_{C\setminus h_i}$. That is, we only need no-broken-circuit terms; in particular, A is spanned by a_N for $N \in \mathrm{NBC}_p(\mathcal{M})$.

To show this set is a basis we count. We proved $\dim A_p = |w_p(\mathcal{M})|$. Brylawski proved $|\mathrm{NBC}_p(\mathcal{M})| = |w_p(\mathcal{M})|$. QED ☐

Convenient notation: $\mathcal{M}' = \mathcal{M} \setminus h$, $\mathcal{M}'' = \mathcal{M}/h$. $A' = A(\mathcal{M}')$, etc.

Theorem 0.2. *Assume h is not a coloop in $\mathcal{M}$. The sequence $0 \xrightarrow{i} A'_p \to A_p \xrightarrow{j} A''_{p-1} \to 0$ is exact.*

Proof. Obviously, $ji = 0$, i.e., $\operatorname{Ker} j \subseteq \operatorname{Im} i$.

We know the dimensions:

$$\dim A_p = |w_p(\mathcal{M})|, \quad \dim A'_p = |w_p(\mathcal{M}')|, \quad \dim A''_{p-1} = |w_{p-1}(\mathcal{M}'')|.$$

Standard matroid theory (by Rota): if h is neither a loop nor a coloop in $\mathcal{M}$, then $p_{\mathcal{M}}(\lambda) = p_{\mathcal{M}'}(\lambda) - p_{\mathcal{M}''}(\lambda)$. From this, remembering that $w_p(\mathcal{M})$ is the coefficient of $\lambda^{r(\mathcal{M})-p}$, we can deduce that $w_p(\mathcal{M}) = w_p(\mathcal{M}') - w_{p-1}(\mathcal{M}'')$. Rota's sign theorem then implies that $|w_p(\mathcal{M})| = |w_p(\mathcal{M}')| + |w_{p-1}(\mathcal{M}'')|$. Restating this in terms of A's,

$$\dim A_p = \dim A'_p + \dim A''_{p-1},$$

which gives $\dim \operatorname{Ker} j = \dim A_p - \dim A''_{p-1} = \dim A'_p = \dim \operatorname{Im} i$ and voilà! ☐

These proofs show how important is Orlik's Proposition 3.20, the one that says $\dim A_X = (-1)^{r(X)} \mu(\hat{0}, X)$, from which we inferred that $\dim A_p = |w_p|$.

Corollary 0.3. *The short exact sequence of Theorem 0.2 splits.*

Proof. From algebra, since the outer terms are free $\mathbb{Z}$-modules. ☐

(Thanks to Chris for timely reminders about the dimension law and splitting.)

— Submitted for your approval by Tom; 26, 29 Feb 2020