Gain Graphs and Hyperplane Arrangements
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Lecture 0 (Overview)
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Let K be a field, let $a \in K^\times$, and let h_{ij}^a be the hyperplane in K^n with equation $x_i = ax_j$. Also, define h_i to be the coordinate hyperplane $h_i = 0$. Note that $h_{ij}^a = h_{ji}^{a^{-1}}$. This property leads to the definition of a "gain graph".

Definition. Let Γ be a graph and $\mathfrak{G}$ a group. Orient the edges of Γ, and let $\vec{E}$ be the set of oriented edges. Denote by e^{-1} the edge e with its opposite orientation. A $\mathfrak{G}$-*gain graph* is $\Phi = (\Gamma, \varphi)$, where $\varphi : \vec{E} \to \mathfrak{G}$ satisfies $\varphi(e^{-1}) = \varphi(e)^{-1}$ for all oriented edges e.

The function φ is called a *gain function*, and $\varphi(e)$ is the *gain* of e.

Gain graphs are also allowed to have *half-edges*, incident to a single endpoint. These are not the same as loops: a loop is an ordinary edge with two endpoints that happen to coincide. Half-edges are not assigned a gain.

Note that we distinguish between "directed" edges and "oriented" edges. A directed edge has a single fixed direction, while an oriented edge has a preferred direction for notational purposes, that may be changed as is convenient. This is similar to orientation of, for example, a simplex in topology: one direction is preferred for notation, but the other may still be used.

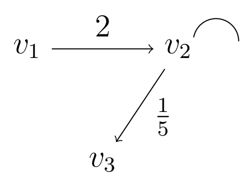
Definition. Let $\Phi = (\Gamma, \varphi)$ be a $\mathfrak{G}$-gain graph, and let W be the walk $v_0 e_1 v_1 \cdots v_{l-1} e_l v_l$ in Γ. We define the *gain of* W to be $\varphi(W) := \varphi(e_1)\varphi(e_2)\cdots\varphi(e_l)$.

In particular, this defines the gain of a circle C. The value of $\varphi(C)$ depends, in general, on both the direction and (if $\mathfrak{G}$ is non-abelian) the initial vertex. However, the property that C has identity gain is independent of both these choices. A circle C whose gain is the identity of $\mathfrak{G}$ is called *balanced* or *neutral*; otherwise, C is *unbalanced* or *non-neutral*. We denote by $\mathcal{B}(\Phi)$ the set of balanced circles of Φ. Finally, Φ is *balanced* or *neutral* when Φ has no unbalanced circles or half-edges (half-edges are considered unbalanced).

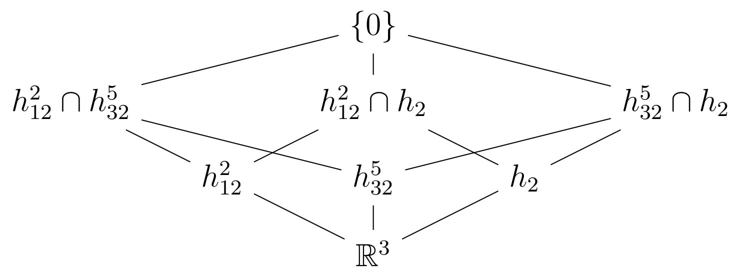
(Though it may make linguistic sense to call an unbalanced gain graph "biased", we reserve that term for a separate notion of a *biased graph*, of which an example is $(\Gamma, \mathcal{B}(\Phi))$. Here, we are singling out a certain collection of circles.)

Suppose $\Phi = (\Gamma, \varphi)$ is a $K^\times$-gain graph with vertex set $V = \{v_1, \ldots, v_n\}$. From Φ we obtain a hyperplane arrangement $\mathcal{A}[\Phi]$ whose hyperplanes are $h_{ij}^{\varphi(e_{ij})}$ for each $e_{ij} \in \vec{E}$, an (oriented) edge between v_i and v_j, and h_i for each half edge e_i, incident to vertex v_i.

Here is an example of a gain graph Φ, with the gains beside the edges and the edges oriented for reading the gains:



Let $K = \mathbb{R}$. As $\varphi(v_1 v_2) = 2$, the hyperplane h_{12}^2 with equation $x_2 = 2x_1$ is in $\mathcal{A}[\Phi]$. Similarly, $h_{23}^{1/5} = h_{32}^5$ and $h_2 \in \mathcal{A}[\Phi]$, where $h_{23}^{1/5}$ has equation $x_3 = \frac{1}{5}x_2$ and h_2 has equation $x_2 = 0$. The intersection lattice is as follows:



Here

$$\begin{aligned} h_{12}^2 \cap h_{32}^5 &= \{(x_1, 2x_1, \tfrac{2}{5}x_1) \mid x_1 \in \mathbb{R}\}, \\ h_{12}^2 \cap h_2 &= \{(0, 0, x_3) \mid x_3 \in \mathbb{R}\}, \\ h_{32}^5 \cap h_2 &= \{(x_1, 0, 0) \mid x_1 \in \mathbb{R}\}. \end{aligned}$$

One may form a "completion" of Φ by adding implied hyperplanes. For example, given the equations $x_1 = 3x_2$, $x_2 = 2x_3$, and $x_3 = -x_1$, we can see that $x_1 = 6x_3 = -6x_1$, which implies $x_1 = 0$ as well as $x_2 = 0$ and $x_3 = 0$, so one may consider the arrangement to be completed by adding in these hyperplanes (as well as others implied by those equations, a large number if K is a large field). This would correspond to adding half-edges incident to each vertex in the gain graph (and edges e_{ij} with all possible gains). I will discuss this notion of completion later in terms of closure within a given gain graph.