Gain Graphs and Hyperplane Arrangements
Lecture 1
Lecture notes transcribed by Nicholas Lacasse

We begin with some preliminary notation and definitions. We use $\mathfrak{G}$ to denote a group and ϵ to denote its identity element. We will typically use Γ for a graph. The reader may be familiar with graphs that have loops and multiple edges (each of which have two ends, each incident with one vertex), which we call *ordinary edges*; but it is less likely that the reader has encountered half-edges or loose edges. A *half-edge* is an edge with one end, which is incident to one vertex. A loose edge is an edge with no ends and no incident vertices.

For a graph Γ we denote its edge set by E and its vertex set by V.

Definition 1. A *gain graph*, Φ, is a pair (Γ, φ) where Γ is a graph and $\varphi : \vec{E} \to \mathfrak{G}$ is a function from the oriented ordinary edges of Γ to a group $\mathfrak{G}$ that satisfies $\varphi(e^{-1}) = \varphi(e)^{-1}$, where e^{-1} denotes e in the opposite orientation. We call φ a *gain function*.

In a gain graph, each edge (with the exception of loose edges) has two possible orientations. Below we show the two possible orientations on links, loops, and half edges and we show the unoriented loose edge.

edge_orientations.jpg

These orientations matter only in order to define the value of the gain of an edge, which is inverted by reversing orientation. We need an arbitrary orientation so that we may define ϕ. In light of this, we define the set $\vec{E}$ to be the set of (arbitrarily) *oriented* edges of Γ. Thus when we write e^{-1} we mean the opposite direction of the arbitrarily chosen orientation. Note that these orientations are not fixed, so we do not have a directed graph.

Consider a gain graph $\Phi = (\Gamma, \varphi)$ and let $W = v_0 e_1 v_1 e_2 \cdots e_l v_l$ be a walk in Γ. We define the *gain of* W to be $\varphi(W) := \varphi(e_1)\varphi(e_2)\cdots\varphi(e_l)$. A *circle* is a 2-regular connected graph (or its edge set). A circle C has a gain $\varphi(C)$ which may depend on the initial point and direction; e.g., if $C = e_1 e_2 \cdots e_k$ then we could also write C as $e_3^{-1} e_2^{-1} e_1^{-1} e_k^{-1} e_{k-1}^{-1} \cdots e_4^{-1}$, and there is no guarantee that $\varphi(e_1 e_2 \cdots e_k) = \varphi(e_3^{-1} e_2^{-1} e_1^{-1} e_k^{-1} e_{k-1}^{-1} \cdots e_4^{-1})$. However, if $\varphi(C) = \epsilon$, then $\varphi(C) = \epsilon$ for any initial point and direction.

Proposition 2. *Let C be a circle in a gain graph Φ. If $\varphi(C) = \epsilon$, then $\varphi(C)$ is independent of which vertex we start at and in which direction we traverse the circle.*

We are most interested in which circles have $\varphi(C) = \epsilon$ and which do not. Thus, by proposition 2, we are justified in writing $\varphi(C)$.

Definition 3. Let $\mathcal{B}(\Phi) := \{\text{circles with gain } \epsilon\}$. For $C \in \mathcal{B}(\Phi)$, we say C is *balanced* or *neutral*.

Definition 4. We call Φ *balanced* (or *neutral*) if every circle of Φ is balanced and Φ has no half edges.

We also call a subgraph of Φ, or an edge set, balanced if every circle in it has gain ϵ and it has no half-edges.

Definition 5. A theta graph is the union of 3 internally disjoint paths with the same two end points.

Definition 6. We denote the free group on a set E by $\mathfrak{F}(E)$.

A gain function φ defines a homormophism $\varphi : \mathfrak{F}(E) \to \mathfrak{G}$ in the "obvious" way. That is, $\varphi(e_1^{\pm 1} e_2^{\pm 1} \cdots e_l^{\pm 1}) = \varphi(e_1)^{\pm 1} \varphi(e_2)^{\pm 1} \cdots \varphi(e_l)^{\pm 1}$.

Proposition 7. $\mathcal{B}(\Phi)$ *has the property that no theta subgraph contains exactly two balanced circles.*

Proof. Let P_1, P_2, P_3 be three internally disjoint paths from v to w. This subgraph has three circles, $P_1 P_2^{-1}, P_2 P_3^{-1}$, and $P_1 P_3^{-1}$. Since φ defines a homomorphism on $\mathfrak{F}(E)$ to $\mathfrak{G}$ and $P_1 P_3^{-1} = (P_1 P_2^{-1})(P_2 P_3^{-1})$, then $\varphi(P_1 P_3^{-1}) = \varphi(P_1 P_2^{-1}) \varphi(P_2 P_3^{-1})$. Suppose without loss of generality that circles $P_1 P_2^{-1}$ and $P_2 P_3^{-1}$ are balanced, i.e., $\phi(P_1 P_2^{-1}) = \phi(P_2 P_3^{-1}) = \epsilon$. Then $\varphi(P_1 P_3^{-1}) = \varphi(P_1 P_2^{-1}) \varphi(P_2 P_3^{-1}) = \epsilon\epsilon = \epsilon$. □

Definition 8. Let $C(\Gamma)$ be the class of all circles in Φ. Let $\mathcal{B} \subseteq C(\Gamma)$. We call $\mathcal{B}$ a *linear class* if no theta subgraph of Γ contains exactly two circles in $\mathcal{B}$.

Definition 9. Let Γ be a graph and let $\mathcal{B}$ be a linear class of circles. We call $(\Gamma, \mathcal{B})$ a *biased graph*.

All the matroid theory of gain graphs will generalize to biased graphs. However, the proofs may be more complicated for biased graphs, because gains provide helpful extra structure beyond the balanced circle class. A good example of this is switching.

Definition 10. *Switching* $\Phi = (\Gamma, \varphi)$ means taking a function $\zeta : V \to \mathfrak{G}$ and replacing φ by φ^ζ, which is defined by $\varphi^\zeta(e_{uv}) = \zeta(u)^{-1}\varphi(e_{uv})\zeta(v)$. By Φ^ζ we mean the gain graph (Γ, φ^ζ) obtained by switching Φ by ζ.

Proposition 11. *For a walk W from u to v, $\varphi^\zeta(W) = \zeta(u)^{-1}\varphi(W)\zeta(v)$. For a closed walk W, $\varphi^\zeta(W)$ is the conjugate of $\varphi(W)$ by $\zeta(u)$.*

Proposition 12. *A circle is balanced in Φ^ζ if and only if it is balanced in Φ. I.e.,*

$$\mathcal{B}(\Phi^\zeta) = \mathcal{B}(\Phi).$$

Here is a use for switching:

Proposition 13. *Let $\Phi = (\Gamma, \varphi)$. Let F be a forest in Γ. Let $\tau : F \to \mathfrak{G}$ be any function. Then we can switch φ so that $\varphi^\zeta|_F = \tau$.*

Proof. Exercise! □

In particular, we often want to pick a maximal forest F of Φ and switch φ so that it is the identity on F.