GAIN GRAPHS AND HYPERPLANE ARRANGEMENTS
LECTURE 10: EXAMPLES!

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In this lecture, we use the results we have developed to explore various examples.

Definition 1. Let $\mathfrak{G}$ be a group and Γ a simple graph. Then $\mathfrak{G}\Gamma = (V(\Gamma), \mathfrak{G} \times E(\Gamma))$ with $\varphi(g, e_{ij}) = g$ is a gain graph called a *group expansion* of Γ, specifically the $\mathfrak{G}$-*expansion*.

The *full* $\mathfrak{G}$-*expansion* $\mathfrak{G}\Gamma^\bullet$ is $\mathfrak{G}\Gamma$ with a half edge added at every vertex.

See Figure 1 for the $\mathfrak{G}$-expansion of a link e.

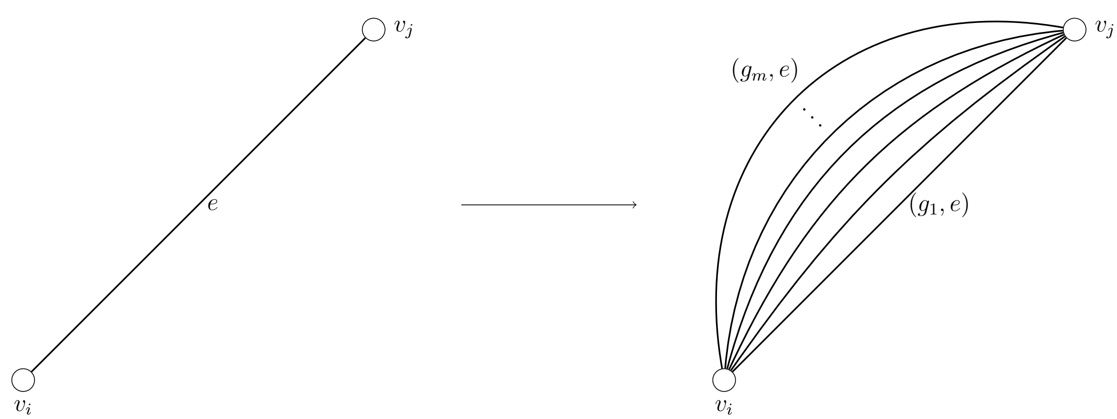
Theorem 2. *Let $\mathfrak{G}$ be a finite group of order m and Γ a simple graph on n vertices. Then*

$$\chi^b_{\mathfrak{G}\Gamma}(\lambda) = m^n \chi_\Gamma\Big(\frac{\lambda}{m}\Big)$$

and

$$\chi_{\mathfrak{G}\Gamma^\bullet}(\lambda) = m^n \chi_\Gamma\Big(\frac{\lambda - 1}{m}\Big).$$

FIGURE 1. On the left is the edge e in Γ. On the right is the set of edges which places e in $\mathfrak{G}\Gamma$, one edge for each element of $\mathfrak{G}$.



Proof. For the first part of the theorem, we start with a 0-free k-coloration γ of $\mathfrak{G}\Gamma$. This coloration is proper when, for all $f \in \mathfrak{G}$, $\gamma(v_j) \neq \gamma(v_i)\varphi(f, e_{ij})$. Suppose $\gamma(v_j) = (h, b)$ and $\gamma(v_i) = (g, a)$. Then $\gamma(v_i)\varphi(f, e_{ij}) = (g, a)\dot{f} = (gf, a)$. So the propriety condition on γ is that $(h, b) \neq (gf, a)$ for all $f \in \mathfrak{G}$. This can only be satisfied if $a \neq b$. We can therefore express the 0-free proper k-coloration $\gamma : V \to \mathfrak{G} \times E$ as $\gamma = (\gamma_{\mathfrak{G}}, \gamma_E)$ where γ_E is a proper k-coloration of Γ and $\gamma_{\mathfrak{G}} : V \to \mathfrak{G}$ is arbitrary. The number of proper k-colorations Γ_E is $\chi_\Gamma(k)$, so the number of 0-free proper k-colorations of $\mathfrak{G}\Gamma$ is $m^n\chi_\Gamma(k)$. Since $\lambda = mk$, we deduce that $\chi^b_{\mathfrak{G}\Gamma}(\lambda) = m^n\chi_\Gamma(\lambda/m)$. This proves the first part of the theorem.

For the second part we put in the half edges to make the graph full. Then the color 0 is excluded, so $\chi_{\mathfrak{G}\Gamma^\bullet}(\lambda) = \chi^b_{\mathfrak{G}\Gamma}(\lambda - 1)$. But $\lambda = km + 1$. So $k = \frac{\lambda-1}{m}$, giving the result. $\square$

Example 3. Consider $\mathfrak{G}K_n^\bullet$. Since $\chi_{K_n}(k) = (k)_n = k(k-1)\cdots(k - [n-1])$, we get the chromatic polynomial formula

$$\begin{aligned}
\chi_{\mathfrak{G}K_n^\bullet}(\lambda) &= m^n\Big(\frac{\lambda - 1}{m}\Big)_n \\
&= m^n\Big(\frac{\lambda - 1}{m}\Big)\Big(\frac{\lambda - 1}{m} - 1\Big)\cdots\Big(\frac{\lambda - 1}{m} - [n-1]\Big) \\
&= (\lambda - 1)(\lambda - 1 - m)(\lambda - 1 - 2m)\cdots(\lambda - 1 - [n-1]m).
\end{aligned}$$

Example 4. Now suppose $\mathfrak{G} \leq K^\times$—as, for example, the finite cyclic group of order m is the group of m-th roots of unity in $\mathbb{C}$, or the cyclic group of order $q - 1$ is the multiplicative group of the finite field $\mathbb{F}_q$. Then

$$\mathscr{H}[\mathfrak{G}K_n^\bullet] = \{h_i : x_i = 0\} \cup \{h_{ij}^g : x_j = x_i g \mid g \in \mathfrak{G}\}.$$

By Theorem **??**, the characteristic polynomial of the arrangement is the chromatic polynomial of $\mathfrak{G}K_n^\bullet$, so

$$p_{\mathscr{H}[\mathfrak{G}K_n^\bullet]}(\lambda) = (\lambda - 1)(\lambda - 1 - m)(\lambda - 1 - 2m)\cdots(\lambda - 1 - [n-1]m).$$

From this we get a formula for the number of regions of the arrangement:

$$\begin{aligned}
(-1)^n p_{\mathscr{H}[\mathfrak{G}K_n^\bullet]}(-1) &= (-1)^n m^n (\frac{-1 - 1m)_n}{=}(-1)^n m^n (\frac{-2}{m})_n \\
&= (2)(2 + m)\ldots(2 + [n-1]m).
\end{aligned}$$

Example 5. Let's apply the preceding examples to the smallest non-trivial group: $\mathfrak{G} = \{\pm 1\}$. Then we are considering the signed graph

$\pm K_n^\bullet$ (short for $\{\pm 1\}K_n^\bullet$). We infer that

$$p_{\mathscr{H}[\pm K_n^\bullet]}(\lambda) = 2^n \left(\frac{\lambda - 1}{2}\right)_n$$

and the number of regions is

$$(-1)^n p_{\mathscr{H}[\pm K_n^\bullet]}(-1) = (-1)^n 2^n \left(\frac{-2}{2}\right)_n = 2^n n!.$$

This has been long known to Lie theorists (who call regions "chambers"), but we have used a different and more general method to get this number.

The connection with Lie theory is historically important, as it was the impetus (stimulated by two questions from Richard Stanley) for the entire theory of gain-graphic matroids and hyperplane arrangements. A *root system* is a finite set of vectors in $\mathbb{R}^n$ that have certain nice integrality properties that I will not state here; they are stated in most books on Lie theory. The indecomposable root systems have been classified; they come in four infinite families, one for each dimension, called the *classical root systems*, and a small number of exceptional root systems. Our interest is in the classical root systems.

Example 6. Let's take our gain group to be $\{\pm 1\} \leq \mathbb{R}^\times$ and let's express the standard basis of $\mathbb{R}^n$ as $b_1, \ldots, b_n$. We associate vectors to the edges of our graph and we associate those vectors to their dual hyperplanes, i.e., the hyperplanes for which they are defining vectors. For an edge e_{ij} we write e_{ij}^+ if it is positive and e_{ij}^- if it is negative. For a half edge we write e_i (as it has no sign). We associate e_{ij}^+ to a vector $\pm(b_j - b_i)$, e_{ij}^- to a vector $\pm(b_j + b_i)$, and e_i to a vector $\pm b_i$. These vectors and their negatives constitute the root system B_n. The vectors $\pm(b_j - b_i)$ determine the hyperplane $x_i = x_j$, the vectors $\pm(b_j + b_i)$ determine the hyperplane $x_i = -x_j$, and the vectors $\pm b_i$ determine the hyperplane $x_i = 0$. These hyperplanes form the root system hyperplane arrangement $\mathscr{B}_n$. If we replace the half edges e_i by negative loops e_{ii}^-, we get vectors $\pm 2b_i$; this results in the root system C_n with hyperplane arrangement $\mathscr{C}_n = \mathscr{B}_n$. If we take only the positive edges, we get the root system called A_{n-1} (because it is not full-dimensional in $\mathbb{R}^n$) and its hyperplane arrangement $\mathscr{A}_{n-1}$. If we take only positive and negative links (i.e., no half edges or loops) we get the root system D_n and the arrangement $\mathscr{D}_n$. These correspond to the four infinite families of root systems.

The root system arrangements are signed-graphic. We list them with their graphs and the number of regions, r, computed from the chromatic polynomials. The circle in $\pm K_n^\circ$ means we add a negative loop instead of a half edge to every vertex (which makes a difference in computing vertex degrees—which we don't do in these notes).

$$\mathscr{A}_{n-1} = \mathscr{H}[K_n] : r = (-1)^n \chi_{K_n}(-1) = (-1)^n(-1)_n = n!,$$
$$\mathscr{B}_n = \mathscr{H}[\pm K_n^\bullet], \mathscr{C}_n = \mathscr{H}[\pm K_n^\circ] : r = (-1)^n \chi_{\pm K_n^\bullet}(-1) = 2^n n!,$$
$$\mathscr{D}_n = \mathscr{H}[\pm K_n] : r = (-1)^n \chi_{\pm K_n}(-1) = 2^{n-1} n!.$$

The last of these needs proof! The tool is the next theorem, which although simple is very convenient for computing chromatic polynomials.

Theorem 7 (Balanced Expansion)**.** *Let Φ be a gain graph without loose edges. Then*

$$\chi_\Phi(\lambda) = \sum_{\substack{W \subseteq V \\ W \, stable}} \chi^b_{\Phi:W^C}(\lambda - 1).$$

Proof. We leave this as an exercise for the reader. $\square$

With Theorem 7 we can observe that

$$\chi_{\pm K_n}(\lambda) = \sum_{W \subseteq V:\ W \text{ stable}} \chi^b_{\pm K_n : W^c}(\lambda - 1).$$

But $W \subseteq V(\pm K_n)$ is stable only when (and when) $W = \emptyset$ or $|W| = 1$, since the gain graph is compete and there are no half-edges or loops. So

$$\begin{aligned}
\chi_{\pm K_n}(\lambda) &= \sum_{\substack{W \subseteq V \\ W \text{ stable}}} \chi^b_{\pm K_n : W^C}(\lambda - 1) \\
&= \chi^b_{\pm K_n}(\lambda - 1) + n\chi^b_{\pm K_n}(\lambda - 1) \\
&= 2^n \chi_{K_n}(\lambda - 1) + n 2^{n-1} \chi_{K_{n-1}}(\lambda - 1) \\
&= (\lambda - 1)(\lambda - 3) \ldots (\lambda - 2n + 1) + n(\lambda - 1)(\lambda - 3) \ldots (\lambda - 2n + 3) \\
&= (\lambda - 1)(\lambda - 3) \ldots (\lambda - 2n + 3) \cdot (\lambda + 1 - n).
\end{aligned}$$

Thus,

$$(-1)^n \chi_{\pm K_n}(-1) = (2)(4) \ldots (2(n-1))(n) = 2^{n-1} n!.$$

That gives the chromatic polynomial and the region count we wanted.

Example 8. By a similar computation (Exercise!), for any group $\mathfrak{G}$ of finite order m,

$$\chi_{\mathfrak{G}K_n}(\lambda) = m^{n-1} \binom{\lambda - 1}{m}_{n-1} [\lambda - (m-1)(n-1)].$$