Lecture 11

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Notetaker: Michael Gottstein

For the gain-graphic arrangements we encountered previously, the gain group was multiplicative: $\mathfrak{G} \leq K^\times$ and the hyperplanes were homogeneous, i.e., subspaces of the vector space K^n. Now we switch to an additive group, $\mathfrak{G} \leq K^+$ to examine a new kind of gain-graphic arrangement, which I call "affinographic" because its hyperplanes are affine translates of graphic hyperplanes. This gives an affine—usually inhomogeneous—arrangement in the affine space $\mathbb{A}^n(K)$. For this type of arrangement we do not use half edges or loose edges.

Definition 1. A hyperplane of the form $x_j = x_i + c$ is called *affinographic*. An *affinographic hyperplane arrangement* is an arrangement whose hyperplanes are affinographic.

Definition 2. Given a gain graph Φ with gain group $\mathfrak{G} \leq K^+$, without half or loose edges, the corresponding affinographic hyperplane arrangement is

$$\mathscr{A}[\Phi] = \{a(e) : e \in E\},$$

where a is a function that gives a hyperplane

$$a(e_{ij}) : x_j - x_i = \varphi(e_{ij}),$$

or equivalently $x_j = x_i + \varphi(e_{ij})$, for each edge of Φ.

Recall from Lecture 7 (Definition **??**) that $\operatorname{Lat^b}\Phi = \{A \in \operatorname{Lat}\Phi : A \text{ is balanced}\}$.

Theorem 3. *Let $S \subseteq E$. Then $\bigcap a(S) \neq \varnothing$ if and only if S is balanced.*

The function a gives a semilattice isomorphism $\operatorname{Lat^b}\Phi \cong \mathscr{L}(\mathscr{A}[\Phi])$.

Proof. We start the proof with three useful lemmas.

Lemma 4. *If C is an unbalanced circle, then $\bigcap a(C) = \varnothing$.*

Proof. Let $C = v_0e_{01}v_1e_{12}v_2 \dots e_{l-1,l}v_l$, where $v_0 = v_l$. Then $x \in \bigcap a(C) \iff x$ satisfies all the equations

$$\begin{aligned} x_1 &= x_0 + \varphi(e_{01}),\\ x_2 &= x_1 + \varphi(e_{12}),\\ &\dots\\ x_l &= x_{l-1} + \varphi(e_{l-1l}), \end{aligned} \tag{1}$$

hence

$$x_l = x_0 + \varphi(e_{01}) + \varphi(e_{12}) + \dots + \varphi(e_{l-1,l}) = x_0 + \varphi(C).$$

But $x_l := x_0$, so this is impossible if $\varphi(C) \neq 0$, i.e., when C is unbalanced. Thus $\bigcap a(C) = \varnothing$. □

Lemma 5. *If $S \subseteq E$ and $F \subseteq S$ is a maximal forest in S, then $\bigcap a(S) = \bigcap a(F)$.*

Proof. For a balanced circle, the equation $x_l = x_{l-1} + \varphi(e_{l-1,l})$ is implied by the others in Equation (1). Indeed, from the first $l-1$ of those equations we infer that $x_{l-1} = x_0 + \varphi(e_{01}e_{12}\dots e_{l-2,l-1}) = x_0 + \varphi(C) - \varphi(e_{l-1,l})$. Since C is balanced and since $x_l = x_0$, this quantity $= x_0 + 0 - \varphi(e_{l-1,l}) = x_l - \varphi(e_{l-1,l})$. Thus, $x_{l-1} = x_l - \varphi(e_{l-1,l})$, which is the desired equation.

This implies that if $x \in \bigcap a(C \setminus e_{l-1,l})$, then $x \in a(e_{l-1.l})$. That is, $\bigcap a(C \setminus e_{l-1,l}) \subseteq a(e_{l-1,l})$.

Now, for edge sets F and S as in the hypothesis, for each $e \in S \setminus F$ there is a circle $C \subseteq F \cup \{e\}$ that contains e. By the preceding calculation, $a(e) \supseteq \bigcap a(C \setminus e) \supseteq \bigcap a(F)$. It follows that $\bigcap_{e \in S \setminus F} a(e) \supseteq \bigcap a(F)$. So, $\bigcap a(S) \supseteq \bigcap a(F)$. As the reverse inclusion is obvious, we have equality. □

Lemma 6. *For a forest $F \subseteq E$, $\bigcap a(F)$ is an affine flat whose codimension is $\#F$.*

Proof. We induct on the number of edges in F.

If there are no edges then the codimension is obviously 0.

A forest with at least one edge has a vertex of degree 1, say v_k with edge e_{mk}. The hyperplane $a(e_{mk})$ is given by the equation $x_k = x_m + \varphi(e_{mk})$, and no other hyperplane in $a(F)$ has x_k in its equation. Consequently, x_k is unrestricted for $x \in \bigcap a(F \setminus e_{mk})$, from which we conclude that

$$a(e_{mk}) \not\supseteq \bigcap a(F \setminus e_{mk})$$

and in $\bigcap a(F)$ we are imposing only the new restriction $x_k = x_m + \varphi(e_{mk})$, from which it follows that

$$a(e_{mk}) \cap \bigcap a(F \setminus e_{mk}) \neq \varnothing.$$

Since $a(e_{mk}) \cap \bigcap a(F \setminus e_{mk}) \neq \varnothing$, the modular law of dimension in $\mathbb{A}^n(K)$ applies; therefore $\operatorname{codim} \bigcap a(F) = \operatorname{codim} \bigcap a(F \setminus e_{mk}) + 1$, so by induction we have the result. □

Now we prove the theorem.

Case 1: S is unbalanced. Then $S \supseteq C$, an unbalanced circle, and $\bigcap a(S) \subseteq \bigcap a(C) = \varnothing$ by Lemma 4.

Case 2: S is balanced. Let F be a maximal forest in S. Then $\bigcap a(S) = \bigcap a(F)$ by Lemma 5, which is not empty by Lemma 6. And $\operatorname{codim} \bigcap a(S) = \operatorname{codim} \bigcap a(F) = \#F$ by Lemma 6.

By elementary graph theory $\#F = n - c(F)$, which $= n - c(S)$ since F is maximal in S. Therefore $\operatorname{rk} \bigcap a(S)$, in $\mathscr{L}(\mathscr{A}[\Phi])$, is equal to $n - c(S) = \operatorname{rk}_\Phi(S)$ (in the frame matroid). Since the ranks match, the closure deduced from $\mathscr{A}[\Phi]$ for balanced edge sets is the same as that in $\mathbf{F}(\Phi)$ for balanced edge sets. This implies that the closed sets in $\mathbf{F}(\Phi)$ that are balanced are in one-to-one correspondence (via the mapping a) with the flats of $\mathscr{A}[\Phi]$. □