Lecture 12: The Lift: Matroid, Examples, and Modular Gains

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Characteristic polynomial. Let's begin with that one of the main theorems about affinographic arrangements—which (at last) gives a solid justificiation for the balanced chromatic polynomial.

Theorem 1. *For a field K, a group $\mathfrak{G} \leq K^+$, and a $\mathfrak{G}$-gain graph Φ, the characteristic polynomial of the affinographic arrangement of Φ is $p_{\mathscr{A}[\Phi]}(\lambda) = \chi^b_\Phi(\lambda)$.*

Proof. For simplicity we write $\mathscr{A} := \mathscr{A}[\Phi]$. Recall that for a balanced subset $S \subseteq E := E(\mathfrak{G})$ and for $\mathscr{S}$ the corresponding set of hyperplanes in $\mathscr{A}$,

$$\dim \bigcap \mathscr{S} = n - \operatorname{rk} \bigcap \mathscr{S} = n - \operatorname{rk}(S) = b(S) = c(S).$$

Then from the polynomial definitions and the balance property in Theorem **??**,

$$p_{\mathscr{A}}(\lambda) = \sum_{\mathscr{S} \subseteq \mathscr{A} \colon \bigcap \mathscr{S} \neq \emptyset} (-1)^{|\mathscr{S}|} \lambda^{\dim(\bigcap \mathscr{S})} = \sum_{S \subseteq E \colon S \text{ balanced}} (-1)^{|S|} \lambda^{b(S)} = \chi^b_\Phi(\lambda). \qquad \square$$

The lift matroid. Now we put the projectivization of an affine arrangement to serious use. We will examine the projectivization of $\mathscr{A}[\Phi]$, written $\mathscr{A}[\Phi]_{\mathbb{P}}$ or $\mathscr{A}_{\mathbb{P}}[\Phi]$, and infer from it a new matroid of a gain graph.

Recall from Stanley's lectures that for an affine hyperplane (or subspace) h in $\mathbb{A}^n(K)$, $h_{\mathbb{P}}$ is its extension into the projective space $\mathbb{P}^n(K)$. For an affine hyperplane arrangment $\mathscr{A}$ in $\mathbb{A}^n(K)$, $\mathscr{A}_{\mathbb{P}} := \{h_{\mathbb{P}} \mid h \in \mathscr{A}\} \cup \{h_\infty\}$, a hyperplane arrangement in $\mathbb{P}^n(K)$.

Note that h_∞, the ideal hyperplane, is isomorphic to $\mathbb{P}^{n-1}(K)$. The arrangement *induced* in h_∞ by $\mathscr{A}_{\mathbb{P}}$ is $\mathscr{A}_{\mathbb{P}}^{h_\infty} := \{h_{\mathbb{P}} \cap h_\infty \mid h \in \mathscr{A}\}$; its matroid is denoted, as usual, by $\mathscr{M}(\mathscr{A}_{\mathbb{P}}^{h_\infty})$. It follows that $\mathscr{M}(\mathscr{A}_{\mathbb{P}}^{h_\infty}) \cong \mathscr{M}(\mathscr{A}_{\mathbb{P}})/h_\infty$, the contraction matroid, by the natural correspondence $h_{\mathbb{P}} \mapsto h_{\mathbb{P}} \cap h_\infty$. Recall also that $h_{\mathbb{P}} \cap h_\infty = h'_{\mathbb{P}} \cap h_\infty$ if and only if h and h' are parallel.

We are going to give an intrinsic characterization of the matroid $\mathbf{L}_0(\Phi)$ implied by the projectivization $\mathscr{A}_{\mathbb{P}}[\Phi]$. The first step is to state that characterization; then we prove it is naturally isomorphic to $\mathscr{M}(\mathscr{A}_{\mathbb{P}}[\Phi])$.

Let $E_0 := E \cup \{e_0\}$, where e_0 is a new object that is not in either E or V. We extend the notion of balance to E_0: we call $S \subset E_0$ *balanced* if $S \subseteq E$ and S is balanced as a subset of E; any other subset of E_0 is *unbalanced*.

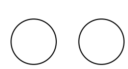
Theorem 2. *For any gain graph Φ, there is a matroid $\mathbf{L}_0(\Phi)$ with ground set E_0 defined by any of the following five equivalent axioms. This definition also applies to any biased graph Ω.*

I. A set $S \subseteq E_0$ is a circuit *of $\mathbf{L}_0$ if and only if it is of one of the following types:*

i. a balanced circle,

ii. a contrabalanced tight handcuff (or tight bracelet),

iii. a contrabalanced loose bracelet,



iv. a contrabalanced theta graph;

v. *$C \cup \{e_0\}$ for an unbalanced circle C.*

II. *A set $S \subseteq E_0$ is an* independent set *of $\mathbf{L}_0$ if and only if it is of one of the following types:*

i. *a forest,*

ii. *an unbalanced unicyclic graph,*

iii. *$F \cup \{e_0\}$ for any forest F.*

III. *The* rank function *of $\mathbf{L}_0$ is*

$$\mathrm{rk}_{\mathbf{L}}(S) = \begin{cases} n - c(S), & \textit{if } S \textit{ is balanced}, \\ n - c(S) + 1, & \textit{otherwise}. \end{cases}$$

IV. *The* lattice of flats *is $\mathrm{Lat}\,\mathbf{L}_0(\Phi) = \mathrm{Lat}^b(\Phi) \cup \mathrm{Lat}_0(\Gamma)$, where $\Gamma = \|\Phi\|$, the underlying graph of Φ, and $\mathbf{L}_0(\Gamma) := \{A \cup \{e_0\} | A \in \mathrm{Lat}(\Gamma)\}$.*

V. *The* closure *of $S \subseteq E_0$ is*

$$\mathrm{cl}(S) = \begin{cases} \mathrm{bcl}(S), & \textit{if } S \textit{ is balanced}, \\ \mathrm{clos}_\Gamma(S) \cup \{e_0\}, & \textit{if } S \subseteq E \textit{ is unbalanced}, \\ \mathrm{clos}_\Gamma(S \setminus \{e_0\}) \cup \{e_0\}, & \textit{if } e_0 \in S. \end{cases}$$

Not a Proof. Sadly, we will not prove this theorem; the proof is too long. We will, however, prove in Theorem 4 that $\mathbf{L}_0$ is the right matroid for $\mathscr{A}_{\mathbb{P}}[\Phi]$, which incidentally proves it is a matroid when Φ has gain group K^+. □

Definition 3 (Lift matroid)**.** The *extended lift matroid* of Φ is the matroid $\mathbf{L}_0(\Phi)$ defined in Theorem 2. The *lift matroid* $\mathbf{L}(\Phi)$ is the restriction of $\mathbf{L}_0(\Phi)$ to the ground set E.

Now, define $a_{\mathbb{P}}$ to be the projective extension to E_0 of the function $a : E \to \mathscr{A}[\Phi]$ defined in Lecture 11. This function satisfies

$$a_{\mathbb{P}}(e) = \begin{cases} h_{\mathbb{P}}, & \text{when } h = a(e) \text{ for some } e \in E, \\ h_\infty, & \text{when } e = e_0. \end{cases}$$

That is, $a_{\mathbb{P}}(e) = a(e)_{\mathbb{P}}$ for an edge e of Φ.

Theorem 4. *The mapping $a_{\mathbb{P}}$ is an isomorphism $\mathbf{L}_0(\Phi) \to \mathscr{M}(\mathscr{A}_{\mathbb{P}}[\Phi])$.*

We give two proofs that share steps but rely on different characterizations of the matroids.

First Proof. For simpler notation we write $\mathscr{A}$ for $\mathscr{A}[\Phi]$, $\mathbf{L}_0 = \mathbf{L}_0(\Phi)$, and $\mathbf{F} = \mathbf{F}(\Phi)$ (the frame matroid). We show $\mathscr{M}(\mathscr{A}_{\mathbb{P}}) \cong_{a_{\mathbb{P}}} \mathbf{L}_0(\Phi)$ by showing the ranks in the two matroids are the same. Let $S \subseteq E_0$ be balanced; then

$$\begin{aligned} \mathrm{rk}_M \bigcap a_{\mathbb{P}}(S) &= \mathrm{codim} \bigcap a_{\mathbb{P}}(S) = \mathrm{codim} \bigcap a(S) = \mathrm{rk} \bigcap a(S) \\ &= \mathrm{rk}_{\mathbf{F}(\Phi)} S = n - b(S) = n - c(S) = \mathrm{rk}_{\mathbf{L}}(S), \end{aligned}$$

where the second equality follows from $\bigcap a(S) \neq \emptyset$ and the fourth from the second part of Theorem **??**.

Suppose S is unbalanced, but $e_0 \notin S$. The graphic hyperplane $h(e_{ij}) : x_i = x_j$ in $\mathbb{A}^n(K)$ has projective extension $h_{\mathbb{P}}(e_{ij})$ in $\mathbb{P}^n(K)$ that satisfies $\{x_j = x_i\}_{\mathbb{P}} \cap h_\infty = \{x_j = x_i + c\}_{\mathbb{P}} \cap h_\infty$ for all $c \in K$ because the hyperplanes are parallel. Hence, $a_{\mathbb{P}}(e) \cap h_\infty = h_{\mathbb{P}}(e) \cap h_\infty$ for any edge $e \in E$. This, in particular, implies that $\mathscr{A}_{\mathbb{P}}^{h_\infty} = \mathscr{H}[\Gamma]_{\mathbb{P}}^{h_\infty}$. Because $\mathscr{H}[\Gamma]$ is homogeneous,

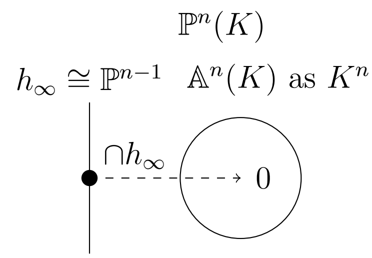


FIGURE 1. The mapping $\cap h_\infty$ inverts to give a unique vector subspace of K^n.

the mapping $\mathscr{H}[\Gamma] \xrightarrow{\cap h_\infty} \mathscr{H}_{\mathbb{P}}[\Gamma]^{h_\infty}$ is a rank-preserving bijection. That bijection implies a matroid isomorphism $\mathscr{M}(\mathscr{H}[\Gamma])_{\mathbb{P}}^{h_\infty} \cong \mathscr{M}(\mathscr{H}[\Gamma])$ (see Figure 1).

Now consider $S = T \cup \{e_0\}$ for some $T \subseteq E$. Then $\bigcap a_{\mathbb{P}}(S) \subseteq a_{\mathbb{P}}(e_0) = h_\infty$, so

$$\bigcap a_{\mathbb{P}}(S) = \bigcap_{e \in T} \big(a_{\mathbb{P}}(e) \cap h_\infty\big) = \bigcap_{e \in T} \big(h_{\mathbb{P}}(e) \cap h_\infty\big) = h_\infty \cap \bigcap_{e \in T} h_{\mathbb{P}}(e_{ij}).$$

Thus the rank of $\bigcap a_{\mathbb{P}}(S)$ is

$$\operatorname{rk} \bigcap a_{\mathbb{P}}(S) = \operatorname{codim} \bigcap a_{\mathbb{P}}(S) = \underbrace{\operatorname{codim} \bigcap h_{\mathbb{P}}(S)}_{n - c(T)} + \underbrace{1}_{h_\infty}. \qquad \square$$

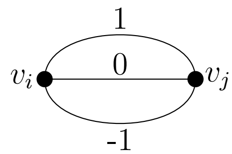
Second Proof. Again we write $\mathscr{A}$ for $\mathscr{A}[\Phi]$. This proof depends on showing that the closed sets of $\mathbf{L}_0(\Phi)$ are the right ones for $\mathscr{M}(\mathscr{A}_{\mathbb{P}})$. For balanced closed sets, this is Theorem **??**.

For unbalanced ones, since they all contain e_0, which corresponds to h_∞, they must correspond to ideal flats of $\mathscr{M}(\mathscr{A}_{\mathbb{P}})$; in other words, subspaces in $\mathscr{L}(\mathscr{A}_{\mathbb{P}}^{h_\infty})$. Such a flat is the intersection with h_∞ of a set of hyperplanes $h_{\mathbb{P}}$ for $h \in \mathscr{S}$ where $\mathscr{S}$ is some subset of $\mathscr{A}$. The affine hyperplane h has equation $x_j - x_i = c$ for a constant c, but its ideal part, $h_{\mathbb{P}} \cap h_\infty$, is independent of c; so we may replace $\mathscr{A}[\Phi]$ by the arrangement $\mathscr{H}[\Gamma]$ of graphic hyperplanes $h(e) : x_j = x_i$ that are parallel to the hyperplanes $a(e)$ of $\mathscr{A}$. Since $\mathscr{H}[\Gamma]$ is homogeneous, the flats $s \in \mathscr{L}(\mathscr{H}_{\mathbb{P}}[\Gamma])$ are determined by their ideal parts $s_\infty := s_{\mathbb{P}} \cap h_\infty$. Therefore, $\mathscr{L}(\mathscr{A}_{\mathbb{P}}^{h_\infty}) \cong \mathscr{L}(\mathscr{H}_{\mathbb{P}}[\Gamma]^{h_\infty}) \cong \mathscr{L}(\mathscr{H}[\Gamma]) \cong \operatorname{Lat}\Gamma \cong \operatorname{Lat}_0\Gamma$ with the third isomorphism given by $a_{\mathbb{P}}$. That proves the ideal flats of $\mathscr{A}_{\mathbb{P}}$ correspond to the unbalanced flats of $\mathbf{L}_0(\Phi)$ via $a_{\mathbb{P}}$. That completes the proof. $\square$

Popular Affinographics. Several affinographic arrangements that have received a lot of attention in recent years are the real affine arrangements of certain *integral gain graphs*—where the gain group is the additive group of integers, $\mathbb{Z}^+$, regarded as a subgroup of $\mathbb{R}^+$. I will describe some of them. In each example I state the gain graph Φ; the arrangement is $\mathscr{A}[\Phi]$.

These graphs are a kind of partial group expansion: expansions of a base graph Δ by subsets of the gain group. To state the gains we assume a special orientation of the base graph: the vertex set is $V = \{v_1, \ldots, v_n\}$ and edges are oriented upwards, i.e., from v_i to v_j where $i < j$; we denote this oriented graph by $\vec{\Delta}$. (Actually, we always use $\vec{K}_n$.) Then, for instance, the notation $\{1, 2, -3\}\vec{\Delta}$ means that each edge e is replaced by three edges with gains 1, 2, and -3 in the upward direction; equivalently, the gains are -1, -2, $+3$ in the downward direction.

Example 5. The *Catalan arrangement* is associated with the gain graph $\{0, \pm1\}\vec{K}_n$. The picture of an expanded edge e_{ij} (with $i < j$) is

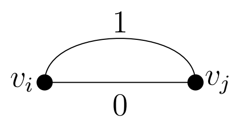


This arrangement gets its name from the curious fact that the number of regions is a Catalan number.

A variation is the *hollow Catalan arrangement*, with gain graph $\{\pm1\}\vec{K}_n$. The picture is the same except that the edge with gain 0 is missing.

A more elaborate variant is the *extended Catalan arrangement*, whose gain graph is $\Phi = \{0, \pm1, \dots, \pm l\}\vec{K}_n$. It has a hollow version as well, without the 0-edges.

Example 6. The *Shi arrangment* has the gain graph $\{0, +1\}\vec{K}_n$. A picture is



The absence of sign symmetry in the gains (i.e., the fact that there is a $+1$ edge e_{ij} but no -1 edge e_{ij}) makes it more difficult to compute the Shi characteristic polynomial than the Catalan arrangement's.

Example 7. The *Linial arrangement*[1] accompanies $\{+1\}\vec{K}_n$.

The Shi and Linial arrangements also have extended variants, though their definitions are not obvious.

We wish to compute the characteristic polynomial $p_{\mathscr{A}}(\lambda)$ of each arrangement in our list, but there is a difficulty: we cannot count proper colorations in an infinite group like $\mathbb{Z}^+$. The solution is to compute $\chi^b_\Phi(\lambda)$ using colors in $\mathbb{Z}^+_m$, which is the additive group of integers modulo m, using the next proposition. For a $\mathbb{Z}^+$-gain graph Φ, define Φ/m to have the same underlying graph and gains modulo m; that is, $\varphi_{\Phi/m}(e) := \varphi_\Phi(e) \mod m$. These are *modular gains.*

Proposition 8. *For a $\mathbb{Z}^+$-gain graph Φ, $\chi^b_\Phi(\lambda) = \chi^b_{\Phi/m}(\lambda)$ if, and only if, m does not divide the gain of any unbalanced circle in Φ.*

Proof. Let $\langle\Phi\rangle$ be the biased graph of Φ. Then $\chi^b_\Phi(\lambda) = \chi^b_{\langle\Phi\rangle}(\lambda)$ and $\chi^b_{\Phi/m}(\lambda) = \chi^b_{\langle\Phi/m\rangle}(\lambda)$ by the definition of χ^b. Also, $\langle\Phi\rangle = \langle\Phi/m\rangle$ if and only if m is not a divisor of the gain of any unbalanced circle, as then unbalanced circles are unchanged by passing from Φ to Φ/m. This implies sufficiency.

For necessity, consider the class $\mathscr{S}$ of unbalanced edge sets that become balanced modulo m. Then

$$(1)\quad \chi^b_{\Phi/m}(\lambda) - \chi^b_\Phi(\lambda) = \sum_{S\in\mathscr{S}} (-1)^{|S|}[\lambda^{b_{\Phi/m}(S)} - \lambda^{b_\Phi(S)}] = \sum_{S\in\mathscr{S}} (-1)^{|S|}\lambda^{b_\Phi(S)}[\lambda^{b_{\Phi/m}(S)-b_\Phi(S)} - 1]$$

[1] Named for Nathan Linial.

Let $b_0 := \min\{b(S) : S \in \mathscr{S}\}$ and $\mathscr{S}_0 := \{S \in \mathscr{S} : b(S) = b_0\}$. Then the term of degree b_0 in Equation (1) has coefficient $-|\mathscr{S}_0|$; that is, the coefficient of λ^{b_0} is reduced by $|\mathscr{S}_0|$ in passing from Φ to Φ/m. This proves that equality fails if any circle becomes balanced upon going to modular gains. $\square$

The modular strategy for computing the balanced chromatic polynomial is to find infinitely many "good" values m, not dividing any circle gain, at which to calculate $\chi^b_\Phi(m)$ by group coloring using the finite cyclic group $\mathbb{Z}_m^+$. We obtain $\chi^b_{\Phi/m}(m)$ by counting proper $\mathbb{Z}_m^+$-colorations, and when m is a good modulus this number equals $\chi^b_\Phi(m)$. Doing this for n good moduli m determines the balanced chromatic polynomial, as we know the degree (n) and the leading coefficient (1). (In practice the same counting procedure succeeds for all $m > \max_{C \in \mathscr{C}(\Phi)} \varphi(C)$ so there is no advantage to restricting to only n moduli.)