Modular Coloring for the Catalan Arrangement

Lecture 13
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Review. We consider affine hyperplane arrangements in $\mathbb{A}^n(\mathbb{R})$, in particular those that arise from an integral gain graph, that is, Φ with the additive gain group $\mathbb{Z}^+$. An edge $e = v_iv_j$ with gain g in Φ gives the hyperplane $x_j - x_i = g$. (Note that this makes the [arbitrary] orientation of the edge significant. If the edge were oriented in the opposite direction with gain g, it would give the hyperplane $x_i - x_j = g$.) The hyperplane arrangement determined by Φ is written $\mathscr{A}[\Phi]$.

The Catalan arrangement $\mathscr{C}_n$ is $\{x_j - x_i = 0, 1, -1 \text{ for } i < j\}$ in $\mathbb{A}^n(\mathbb{R})$ where $\mathbb{A}^n$ denotes n dimensional affine space. The gain graph corresponding to the Catalan arrangement is $\{0, \pm 1\}\vec{K}_n$, that is, K_n with three edges, bearing gains 0, 1, and -1, between each pair of vertices. We call it the *Catalan gain graph*. (Here $\vec{K}_n$ denotes K_n with vertex set $\{v_1, v_2, \ldots, v_n\}$ and all edges oriented upward for the assignment of gains. The same convention can be applied to any graph.) To show why this notation is useful, I mention the Shi arrangement, $\mathscr{S}_n := \mathscr{A}[\{0,1\}\vec{K}_n]$, whose hyperplanes are $x_j - x_i = 0, 1$ for $i < j$.

Our goal is to compute the characteristic polynomial of the Catalan and related arrangements. We achieve this by using three previous theorems. The first two theorems are:

Theorem 1 (Theorem **??**).

$$p_{\mathscr{A}[\Phi]}(\lambda) = \chi^b_\Phi(\lambda) := \sum_{S \subseteq E:\ S \text{ balanced}} (-1)^{\#S} \lambda^{b(S)}.$$

Theorem 2 (Theorem **??** with $k = 1$). *If $\mathfrak{G}$ is a finite group, then $\chi^b_\Phi(\#\mathfrak{G})$ is the number of proper $\mathfrak{G}$-colorations of Φ.*

A proper $\mathfrak{G}$-coloration of Φ is a mapping $\gamma : V(\Phi) \to \mathfrak{G}$ such that, for each edge $e = v_iv_j$, say with gain g, then $\gamma(v_j) \neq \gamma(v_i)g$.

These results let us use coloring methods to determine the characteristic polynomial of an arrangement. However, if our group is, like $\mathbb{Z}^+$, infinite, then so is the number of proper colorations. That creates an obvious difficulty. Fortunately, we have a third theorem to deal with the difficulty. Let $\mathscr{B}(\Phi)$ denote the set of balanced circles of Φ.

Theorem 3 (Definition **??**). *Suppose the underlying graphs of Φ and Φ' are the same and moreover $\mathscr{B}(\Phi) = \mathscr{B}(\Phi')$. Then $\chi^b_\Phi(\lambda) = \chi^b_{\Phi'}(\lambda)$.*

So if we can change the gains on the Catalan gain graph so they are in a finite group, without changing the list of balanced circles, then we may get a meaningful count of proper colorations. This can be done. The idea is to take the integral gains modulo m for $m > n$, changing the gain group from $\mathbb{Z}^+$ to $\mathbb{Z}_m^+$. This will not destroy balance of any circle because if a circle has gain 0 in $\mathbb{Z}$, it has gain 0 modulo m. It will not create new balanced circles because, since the largest magnitude of a gain is 1, no circle has gain larger than n. We formulate this method as a lemma.

Lemma 4. Suppose Φ is an integral gain graph and $m \in \mathbb{Z}_{>0}$ is not the gain of any circle in Φ. Let Φ/m be the same gain graph with gains interpreted modulo m, so the gain group is $\mathbb{Z}_m$. Then $\chi^b_\Phi(\lambda) = \chi^b_{\Phi/m}(\lambda)$ and $\chi_\Phi(\lambda) = \chi_{\Phi/m}(\lambda)$.

Catalan calculations. We are now tasked with counting the number of proper $\mathbb{Z}_m^+$-colorations of $\{0, \pm1\}K_n$. This means we need to count functions $\gamma : V = \{v_1, \dots, v_n\} \to \mathbb{Z}_m^+$ such that $\gamma(v_j) - \gamma(v_i) \neq 0, \pm1$ for all $i \neq j$. We encourage the reader to "close the book" and attempt to work out a solution before continuing.

Here is our class's solution to the coloring problem. We view the vertices v_i as objects that we will be placing into bins. The bins are labeled with integers from 0 to $m - 1$. No two vertices may be placed in the same bin, so there will be $m - n$ empty bins. Let us label the empty bins with the integers from 0 to $m - n - 1$. Now fix vertex v_1 in the space to the left of bin 0 and we place the remaining vertices in the spaces between empty bins, at most one to each space. There are $m-n-1$ such spaces and we choose $n-1$ of them for vertices, in $\binom{m-n}{n-1}$ possible ways. Those vertices may be permuted in any order, giving us a factor of $(n - 1)!$. Now we have a sequence of length m that consists of n vertices and $m - n$ empty bins, with v_1 in position 0. Assign each vertex the number that is its position in this sequence; thus each v_i gets a label in $\mathbb{Z}_m$. To allow for the m ways v_1 could be labelled, we can shift the whole pattern cyclically by any amount from 0 to $m-1$. This gives a total number of labellings equal to $m\binom{m-n-1}{n-1}(n - 1)!$. Each labelling is a proper $\mathbb{Z}_m^+$-coloration of $\{0, \pm1\}K_n/m$ and we obtain every such proper coloration.

Let C_n denote the n-th Catalan number: $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Theorem 5. *For the Catalan arrangement $\mathscr{C}_n$:*
(1) $p_{\mathscr{C}_n}(\lambda) = \lambda(\lambda - n - 1)_{n-1}$.
(2) *$\mathscr{C}_n$ has $n!C_n$ regions.*

Proof. By Theorem 2 we have found the balanced chromatic polynomial of $\{0, \pm1\}K_n/m$. The first conclusion follows by Theorems 3 and 1.

The second part follows, according to Theorem **??**, by calculating $(-1)^n p_{\mathscr{C}_n}(-1)$. □

Related to Catalan. Here is a closely related arrangement, with a nice exercise.

Example 6. The *hollow Catalan arrangement* is $\mathscr{C}_n^o = \mathscr{A}[\{\pm 1\}K_n]$. That is, it is the Catalan arrangement without the graphic hyperplanes $x_i = x_j$.

Calculate the characteristic polynomial $p_{\mathscr{C}_n^o}(\lambda)$ and the number of regions of $\mathscr{C}_n^o$.

And here is another related arrangement with (naturally) another exercise.

Example 7. The *extended Catalan arrangement* for a positive integer k is

$$\mathscr{C}_{n,k} = \mathscr{A}[\{0, \pm 1, \pm 2, \ldots, \pm k\}K_n].$$

There is also the hollow extended Catalan arrangement, $\mathscr{C}_{n,k}^{\circ}$, whose definition is obvious.

Calculate the characteristic polynomial $p_{\mathscr{C}_{n,k}}(\lambda)$ and the number of regions of $\mathscr{C}_{n,k}$.