# Lecture 15: ???

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This proof of Corollary **??** cannot work for the Shi arrangement $\mathscr{S}_n$ because for each pair of coordinates $(i, j)$ with $i < j$ there is a hyperplane $x_j - x_i = 1$ but no hyperplane $x_i - x_j = 1$. Thus, a transposition does not preserve the Shi arrangement and the image of a Shi region is not a region any more. Nonetheless, the Shi regions have other interesting combinatorics. Since as we have seen

$$p_{\mathscr{S}_n}(\lambda) = \lambda(\lambda - n)^{n-1},$$

thus $r(\mathscr{S}_n]) = (n+1)^{n-1}$, which is well known (since Cayley) to be the number of spanning trees of $K_{n+1}$. This coincidence naturally invites a combinatorist to seek a proof by bijection—and it has been done.

**Example: Root system arrangements.** A main example of two-term hyperplane arrangements $\mathscr{H}[\Phi]$ is arrangements obtained from the classical root systems (already introduced in Lecture 10). Root systems originated in Lie algebra but they have turned out to be widely interesting, including in combinatorics. Stanley [**?**, Section 5.1] defines the main ones in terms of vectors and dual hyperplanes, but I will define them in terms of vectors and gain graphs over the 2-element gain group $\{\pm\}$, i.e., signed graphs.

There are four infinite sequences of *classical root systems*, written $A_{n-1}$, $B_n$, $C_n$, and $D_n$, each one naturally described in $\mathbb{R}^n$, and there are also finitely many *exceptional root systems*, which do not fit well with gain graphs so I will ignore them. To describe the classical ones, I write $b_i$ for the $i$th standard unit vector in $\mathbb{R}^n$. The linear-algebra dual to a vector $v$ is the hyperplane $\{x \in \mathbb{R}^n : v \cdot x = 0\}$.

(1) $A_{n-1} = \{b_i - b_j : i \neq j\}$. The dual arrangement is $\mathscr{A}_{n-1} = \mathscr{H}[+K_n]$, as you can easily verify.
(2) $D_n = A_{n-1} \cup \{\pm(b_i + b_j) : i \neq j\}$. The dual arrangement is $_n = \mathscr{H}[\pm K_n]$.
(3) $B_n = D_n \cup \{b_i : i \leq n\}$. The dual arrangement is $_n = \mathscr{H}[\pm K_n']$, where the prime denotes a half edge at every vertex. (A half edge has degree 1 and has no gain.)
(4) $C_n = C_n \cup \{2b_i : i \leq n\}$. The dual arrangement is $\mathscr{C}_n = \mathscr{H}[\pm K_n^\circ]$, where the superscript denotes a negative loop at every vertex. While $C_n \neq B_n$, the duals are the same: $_n = \mathscr{C}_n$. (Do not confuse this with the Catalan arrangement.)

We met these arrangements in Lecture 10, Example **??** et seq., but at that time I didn't mention the root systems themselves. I add to this list the *threshold arrangement*:

(5) $T_n = \{\pm(b_i + b_j) : i \neq j\}$ and the threshold arrangement, which is the dual arrangement $_n = \mathscr{H}[-K_n]$.

There are many questions related to root systems of the following kind:

Generalize a construction or property from affinographic arrangements (like the Catalan and Shi arrangements) to similar affine perturbations of $_n$, or possibly $_n$. (The main interest is in $_n$.) For instance, there are Type B Catalan and Shi arrangements. One question, of course, is the characteristic polynomials. Another is to describe the dissection of the fundamental region of the corresponding root system arrangement, as with the Catalan and Shi arrangements in relation to $\mathscr{A}_{n-1}$. The same questions can be asked for Catalan and Shi threshold arrangements, on which there has been some work: see [**?**, **?**].