Gain Graphs and Hyperplane Arrangements

Lecture 16: Exponential Sequences

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Notetaker: Andrew Lamoureux

We consider a remarkable property of certain sequences of arrangements of increasing dimensionality and its interpretation in terms of gain graphs. We start by copying Stanley:

Definition 1 (Stanley's definition)**.** Let $\mathbf{F}$ be a field. A sequence $(\mathscr{A}_n \mid n \geq 1)$ of hyperplane arrangements is an *exponential sequence of arrangements* if it satisfies:

(S1) each $\mathscr{A}_n$ is an affine hyperplane arrangement in $\mathbb{A}^n(\mathbf{F})$;
(S2) each $\mathscr{A}_n$ is affinographic (i.e., each hyperplane is an affine translate of a graphic hyperplane); and
(S3) for each n and $B \subseteq [n]$, the hyperplanes with coordinates in B yield an arrangement $\mathscr{A}_n{:}B$ such that $\mathscr{L}(\mathscr{A}_n{:}B) \cong \mathscr{L}(\mathscr{A}_{|B|})$.

I want to impose a stronger axiom than (S3). (A1, A2) are the same but (A3) is more restrictive.

Definition 2 (Our definition)**.** A sequence $(\mathscr{A}_n \mid n \geq 1)$ of hyperplane arrangements is an *exponential sequence of arrangements* if it satisfies:

(A1) each $\mathscr{A}_n$ is an affinographic hyperplane arrangement $\mathscr{A}[\Phi_n]$ in $\mathbb{A}^n(\mathbf{F})$;
(A2) each Φ_n is an $\mathbf{F}^+$-gain graph of order n with $V(\Phi_n) = \{v_i \mid i \in [n]\}$; and
(A3) for each n and $B \subseteq [n]$, the induced subgraph $\Phi_n{:}B \cong \Phi_{|B|}$ (which implies (3)).

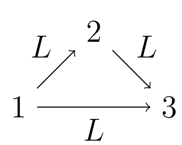
Viewing this in terms of $\mathbf{F}^+$-gain graphs leads to a simple generalization.

Definition 3. Let $\mathfrak{G}$ be a group. A sequence $(\Phi_n \mid n \geq 1)$ of gain graphs is an *exponential sequence of gain graphs* if it satisfies:

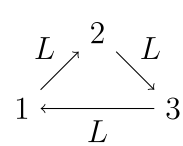
(G1) each Φ_n is a finite $\mathfrak{G}$-gain graph of order n with $V(\Phi_n) = \{v_i \mid i \in [n]\}$; and
(G2) for each n and $B \subseteq V(\Phi_n)]$, the induced subgraph $\Phi_n{:}B \cong \Phi_{|B|}$.

Let's examine (A3) (equivalantly, (G2)) carefully. For $|B| = 1$, it says nothing. For $|B| = 2$, by definition $\Phi_2 = L\vec{K}_2$ for some finite $L \subseteq \mathbf{F}$. Hence for all $v_i, v_j \in V(\Phi_n)$, $\Phi_n{:}\{v_i, v_j\} \cong L\vec{K}_2$, and this isomorphism is natural in the sense that it preserves structure (not necessarily in the sense of category theory). Recall that an isomorphism of gain graphs $\alpha : \Phi \to \Phi'$ is an isomorphism $\|\Phi\| \to \|\Phi'\|$ of underlying graphs such that $\varphi(e) = \varphi(e^\alpha)$ for every edge e.

Now consider $|B| = 3$. The gain graph Φ_3 must be one of the following (up to permuting vertices), with the arrows showing the sense in which to read the gains in L:

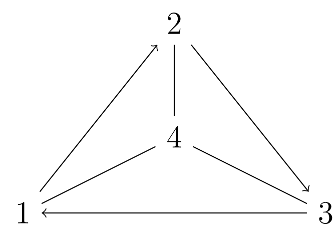


transitive



cyclic

If L is sign-symmetric, i.e., $L = -L$, there is no difference between these two possibilities. Otherwise, there is, for $n \geq 3$: every edge $v_i v_j$ of K_n has a preferred orientation (the one in which the gain set is L, not $-L$) and every induced subgraph of order 3 of every Φ_n for $n > 3$ is of the same kind: all are transitive, or all cyclic. But if all are cyclic, we have a failure at Φ_4. Suppose the outer triangle 123 in Φ_4 is cyclic, as in the following picture:



To make $\triangle 124$ cyclic, edge 14 must be oriented $1 \to 4$, but to make $\triangle 143$ cyclic, edge 14 must be oriented $4 \to 1$. Hence, Φ_4 cannot be oriented cyclically. The conclusion is:

Proposition 4. *An exponential sequence of gain graphs satisfies $\Phi_n \cong L\vec{K}_n$ for some finite subset $L \subseteq \mathfrak{G}$.*

An exponential sequence of arrangements (per Definition 2) satisfies $\mathscr{A}_n \cong \mathscr{A}[L\vec{K}_n]$ for some finite subset $L \subseteq \mathbf{F}^+$.

Proof. Suppose L is not sign-symmetric (otherwise, this is clear). We show that $V(K_n) = \{v_i \mid i \in [n]\}$ can be totally ordered so as to prove the theorem. Define $v_i < v_j$ when the function $v_i \mapsto v_1$, $v_j \mapsto v_2$ is an isomorphism $\Phi_n{:}\{v_i, v_j\} \cong \Phi_2$. This is a strict total order:

(1) (Irreflexivity) There is no bijection $\{v_i\} \to \{v_1, v_2\}$.
(2) (Anti-symmetry) Since $-L \neq L$, it can't be the case that both bijections $\{v_i, v_j\} \to \{v_1, v_2\}$ induce gain-graph isomorphisms; one must give reversed direction of gains.
(3) (Totality) If $v_i \not< v_j$, then the opposite function, $v_i \mapsto v_2$ and $v_j \mapsto v_1$, must be an isomorphism.
(4) (Transitivity) From the analysis of Φ_4 we know that $v_i < v_j < v_k < v_i$ is impossible, so by totality, $v_i < v_j < v_k$ implies $v_k < v_i$.

We have proved that $\Phi_n \cong L\vec{K}_n$ by a permutation of the vertex set, namely, the permutation that carries the ordering $v_{i_1} < v_{i_2} < \cdots < v_{i_n}$ of $V(\vec{K}_n)$ to the natural ordering $v_1 < v_2 < \cdots < v_n$. □

Example 5 (A counterexample)**.** To see that (A3) truly gives a different definition from (S3), consider the following example. Let $\mathfrak{A}$ be an infinite group with an element g of infinite order. Define $\Phi_n := g\vec{K}_n$ for $n \leq 100$ and $\Phi_n := (2g)\vec{K}_n$ for $n > 100$. Then the biased graphs satisfy $\langle \Phi_n \rangle = (K_n, \emptyset)$ for all n, hence $\mathrm{Latb}(\Phi_n{:}B) \cong \mathrm{Latb}(\Phi_{|B|}$ for every $B \subseteq V(\Phi_n)$, whence $\mathscr{L}(\mathscr{A}_n{:}B) \cong \mathscr{L}(\mathscr{A}_{|B|}$ for every B; yet $\Phi_n{:}\{v_i, v_j\} \neq \Phi_2$ for $n > 100$.

Exercise 6. Prove the statement about the biased graph.

The difference between (S3) and (A3) should not bother us. I expect that in every case of interest, it is the underlying gain graphs that are isomorphic.

Now we have the surprising main theorem. To simplify the notation I will write $p_n(t)$ for $p_{\mathscr{A}_n}(t)$ and so forth.

Theorem 7 (Stanley, Theorem 5.17)**.** *If* $p_n(t) := p_{\mathscr{A}_n}(t)$*, then*

$$\sum_{n=0}^{\infty} p_n(t)\frac{x^n}{n!} = \Big(\sum_{n=0}^{\infty} p_n(-1)\frac{x^n}{n!}\Big)^{-t}.$$

The expression $\sum_{n=0}^{\infty} p_n(t)\frac{x^n}{n!}$ is the exponential generating function for the sequence $p_n(t)$. Stanley writes $(-1)^n r_n$ instead of $p_n(-1)$ because $r_n := r(\mathscr{A}_n)$ is the number of regions of the arrangement, obviously a number of particular interest. I will adopt that notation in examples in the next lecture. But for now, I wish to rewrite his proof in terms of gain graphs. Thus, I assume we have an exponential sequence of gain graphs Φ_n with balanced chromatic polynomials $\chi_n^b(t) := \chi_{\Phi_n}^b(t)$. We assume that $p_0(t) = \chi_0^b(t) = 1$.

Proof. The theorem can be rewritten as

$$\text{(1)} \qquad \text{LHS} = \sum_{n\geq 0} \chi_n^b(t)\frac{x^n}{n!} = \Big(\sum_{n\geq 0} \chi_n^b(-1)\frac{x^n}{n!}\Big)^{-t} = \text{RHS}.$$

Our proof takes advantage of the classic *exponential formula*:

$$\sum_{n\geq 0}\frac{x^n}{n!}\sum_{\text{order-}n\text{ objects}O} f(O) = \exp\Big(\sum_{n\geq 1}\frac{x^n}{n!}\sum_{\text{order-}n\text{ connected objects}O} f(O)\Big),$$

provided that f is a function such that $f(O) =$ the product of $f(O')$ over all connected components O' of O. For example, the left side may count the number of n-vertex forests while the right side exponent counts the number of n-vertex trees; the left side may count the number of 2-regular graphs of order n and the right side would count the number of n-vertex circles. (For counting, $f(O) = 1$.) In our case, we are interested in the balanced closed sets in $L\vec{K}_n$ on the left and the connected balanced closed sets in $L\vec{K}_n$ on the right, and instead of counting we are using the balanced chromatic polynomial. Did I mention that

$$\chi_{\Phi_1\cup\Phi_2}(\lambda) = \chi_{\Phi_1}(\lambda)\chi_{\Phi_2}(\lambda) \text{ and } \chi^b_{\Phi_1\cup\Phi_2}(\lambda) = \chi^b_{\Phi_1}(\lambda)\chi^b_{\Phi_2}(\lambda)?$$

These formulas follow easily from the definitions and the facts that the size and number of balanced components of an edge set of the disjoint union are additive:

$$|S| = |S\cap E_1| + |S\cap E_2| \text{ and } b(S) = b(S\cap E_1) + b(S\cap E_2).$$

We know that in (1),

$$\text{LHS} = \sum_{n\geq 0}\sum_{S\in\text{Lat}^b\,\Phi_n}\mu(\emptyset,S)t^{n-\text{rk}(S)}\frac{x^n}{n!} = \sum_{n\geq 0}\sum_{S\in\text{Lat}^b\,\Phi_n}\mu(\emptyset,S)t^{b(S)}\frac{x^n}{n!}.$$

Let S have the balanced components $S_1,\dots,S_k$. They yield a partition $\pi(S)$ of $V_n := V(\Phi_n)$ into the subsets $V(S_1),\dots,V(S_n)$. Then $\mu(\emptyset,S) = \prod_{i=1}^k \mu(\emptyset,S_i)$ because the interval from $\emptyset$ to S is a product: $[\emptyset,S] \cong \prod_{i=1}^k[\emptyset,S_i]$ (Exercise!). We can rewrite the Möbius product in terms of $\pi(S)$: $\mu(\emptyset,S)t^{b(S)} = \mu(\emptyset,S)t^k = \prod_{i=1}^k \mu(\emptyset,S_i)t$. Since $S_i = S{:}V(S_i)$ and $\pi(S) =$

$\{V(S_1), \ldots, V(S_k)\}$,

$$\sum_{S\in \mathrm{Lat}^b \Phi_n} \mu(\emptyset, S) t^{b(S)} = \sum_{S\in \mathrm{Lat}^b \Phi_n} \prod_{i=1}^{k} \mu(\emptyset, S_i) t = \sum_{S} \prod_{B\in\pi(S)} \mu(\emptyset, S{:}B) t$$
$$= \sum_{\pi\in\Pi_n} \prod_{B\in\pi} \Big(\sum_{S:\,\pi(S)=\pi} \mu(\emptyset, S{:}B) t \Big) = \sum_{\pi} \prod_{B\in\pi} \tilde{\chi}_{|B|}(t),$$

where we define the convenient notation

$$\tilde{\chi}_n(t) := \sum_{S\in \mathrm{Lat}^b \Phi_n:\,\pi(S)=\{[n]\}} \mu(\emptyset, S) t^{b(S)} = \sum_{S\in \mathrm{Lat}^b \Phi_n:\,\pi(S)=\{[n]\}} \mu(\emptyset, S) t$$

because $b(S) = 1$ if $\pi(S) = \{[n]\}$.

Now we rewrite the left side in (1) using magic:

$$\mathrm{LHS} = \sum_{n\geq 0} \chi^b_n(t) \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{\pi} \prod_{B\in\pi} \tilde{\chi}_{|B|}(t) t^{b(S)} \frac{x^n}{n!} = \exp\Big(\sum_{n\geq 1} \tilde{\chi}_n(t) \frac{x^n}{n!} \Big)$$
$$= \exp\Big(t \sum_{n\geq 1} \sum_{S\in \mathrm{Lat}^b \Phi_n:\,\pi(S)=\{[n]\}} \mu(\emptyset, S) \frac{x^n}{n!} \Big)$$
$$= \Big[\exp\Big(\sum_{n\geq 1} \sum_{S\in \mathrm{Lat}^b \Phi_n:\,\pi(S)=\{[n]\}} \mu(\emptyset, S) \frac{x^n}{n!} \Big) \Big]^t.$$

We substitute $t = -1$ to get another formula:

$$\sum_{n\geq 0} \chi^b_n(-1) \frac{x^n}{n!} = \exp\Big(\sum_{n\geq 1} \tilde{\chi}_n(-1) \frac{x^n}{n!} \Big) = \Big[\exp\Big(\sum_{n\geq 1} \sum_{S\in \mathrm{Lat}^b \Phi_n:\,\pi(S)=\{[n]\}} \mu(\emptyset, S) \frac{x^n}{n!} \Big) \Big]^{-1}.$$

Compare the last expressions of these two formulas: They are the same except for the exponent. Therefore,

$$\Big(\sum_{n\geq 0} \chi^b_n(-1) \frac{x^n}{n!} \Big)^{-t} = \mathrm{LHS},$$

which proves (1) and the theorem. □

The proof never uses arrangements; it is valid for any exponential sequence of gain graphs. That is, we have a generalization independent of fields.

Theorem 8. *If $(\Phi_n \mid n \geq 0)$ is an exponential sequence of gain graphs, then*

$$\sum_{n=0}^{\infty} \chi^b_n(t) \frac{x^n}{n!} = \Big(\sum_{n=0}^{\infty} \chi^b_n(-1) \frac{x^n}{n!} \Big)^{-t}.$$

(Stanley's Exercise 5.10 is a much more interesting and less expected generalization.)

Example 9. Suppose $\mathfrak{G}$ is a finite group and $\Phi_n = \mathfrak{G}K_n$. Then Theorem 8 applies. In this case we know the balanced chromatic polynomial: it is $|\mathfrak{G}|^n (t/|\mathfrak{G}|)_n$ and the evaluation at -1 is $(|\mathfrak{G}| - n + 1)(|\mathfrak{G}| - 2n + 1) \cdots (1)$.