Lecture 17: More Exponential Sequences; More Gain Expansions

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Let's explore Stanley's Theorem 5.17.

Theorem 1 (Stanley, Theorem 5.17)**.** *Let $(\mathscr{A}_1, \mathscr{A}_2, \ldots)$ be an ESA. Then*

$$\sum_{n\geq 0} \chi_{\mathscr{A}_n}(t)\frac{x^n}{n!} = \left(\sum_{n\geq 0}(-1)^n r(\mathscr{A}_n)\frac{x^n}{n!}\right)^{-t},$$

where $r(\mathscr{A}_n)$ is the number of regions of $\mathscr{A}_n$.

Example 2 (Catalan as exponential sequence)**.** We begin with the Catalan arrangement $\mathscr{C}_n = \mathscr{A}[\{0, \pm 1\}K_n]$ and what we can do for it with Theorem 1. The Catalan generating function has the following well known formula (e.g., see Wikipedia!):

$$C(x) := \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x} = \frac{2}{1+\sqrt{1+4x}}. \tag{1}$$

Using gain graphs we formulated the following balanced chromatic polynomial for the affinographic arrangement $\mathscr{A}[\{\pm 1, 0\}K_n]$, which is the Catalan arrangement $\mathscr{C}_n$. $\chi^b_{\{\pm 1,0\}K_n}(\lambda)$ which we have shown is equal to the characteristic polynomial $p_{\mathscr{C}_n}(\lambda)$. The Catalan sequence $(\mathscr{C}_1, \mathscr{C}_2, \ldots)$ is an ESA, so by Theorem 1, $r(\mathscr{C}_n) = n!C_n$, and Equation (1),

$$\begin{aligned}\sum_{n\geq 0} p_{\mathscr{C}_n}(t)\frac{x^n}{n!} &= \left(\sum_{n\geq 0}(-1)^n n! C_n \frac{x^n}{n!}\right)^{-t} = \left(\sum_{n\geq 0} C_n(-x)^n\right)^{-t} \\ &= \left(\frac{1-\sqrt{1+4x}}{2x}\right)^{-t} = \left(\frac{1+\sqrt{1-4x}}{2}\right)^{t}.\end{aligned}$$

Substituting $t = +1$, the left-hand side is $\sum_{n\geq 0}(-1)^n b(\mathscr{C}_n)\frac{x^n}{n!}$, where $b(\mathscr{C}_n)$ is the number of bounded regions of $\mathscr{C}_n$, so

$$\sum_{n\geq 0} b(\mathscr{C}_n)\frac{(-x)^n}{n!} = \frac{1+\sqrt{1-4x}}{2}.$$

Example 3 (The complete graph as exponential sequence)**.** Let's see Theorem 1 at work in an obvious case. Consider the complete graph arrangements $\mathscr{A}[K_n]$, where $p_n(t) = (t)_n$ and $r_n = n!$. In Theorem 1 the left side is $\sum_{n\geq 0}\binom{t}{n}x^n = (1+x)^t$ by the binomial series, while the right hand side is $\left(\sum_{n\geq 0} n!\frac{(-x)^n}{n!}\right)^{-t}$, which equals $(\frac{1}{1+x})^{-t} = (1+x)^t$ by the geometric series. So the theorem holds here, unsurprisingly.

Example 4 (Shi as exponential sequence)**.** Now a not-so-obvious case, the Shi arrangements $\mathscr{S}_n$. Here $p_n(t) = t(t-n)^{n-1}$ and $r_n = (n+1)^{n-1}$, so by Theorem 1,

$$\sum_{n\geq 0} t(t-n)^{n-1}\frac{x^n}{n!} = \left(\sum_{n\geq 0}(n+1)^{n-1}\frac{-x^n}{n!}\right)^{-t},$$

which is (to put it mildly) a less trivial identity to check.

Arrangements connected to interval orders. An interval order is a partially ordered set that can be represented by intervals $I_i = [a_i, b_i]$ for $i = 1, 2, \ldots, n$ in the real line, with $I_i < I_j \iff b_i < a_j$. See Stanley, Section 5.5, for more about interval orders. Here my interest is in the arrangements he finds in connection with interval orders of a certain kind. The arrangements are those of the following kind.

Take a finite subset $L = \{l_1, l_2, \ldots, l_n\} \subset \mathbb{R}_{>0}$ and set $\eta = (l_1, \ldots, l_n)$. Let $\mathcal{P}_\eta$ denote the set of all interval orders P on $[n]$ such that there exist intervals $I_1, \ldots, I_n$ corresponding to P (with I_i corresponding to $i \in P$) such that $l\,(I_i) = l_i$. In other words, $i < j$ if and only if I_i lies entirely to the left of I_j. We now come to the connection with arrangements. Given $\eta = (l_1, \ldots, l_n)$ as above, define the arrangement $\mathscr{J}_\eta$ in $\mathbb{R}^n$ by letting its hyperplanes be given by $x_j - x_i = l_i,\ i \neq j$. This is the affinographic arrangement $\mathscr{A}[\pm LK_n]$.

Holding L fixed, these form an exponential sequence of affinographic arrangements. We would have trouble applying Theorem **??** because we have no way to compute the characteristic polynomial (that is, the balanced chromatic polynomial $\chi^b_{\pm LK_n}(\lambda)$) or the number of regions. More on this later.

The gain graph $\pm LK_n$ is a kind of gain expansion of K_n, similar to group expansions (Theorem **??**) but not so regular since we expand by a small subset of the gain group $\mathbb{R}^+$. We have no general formula for $\chi^b_{\pm LK_n}(\lambda)$. However, if we can convert the gains l_i to integers, we would have gain group $\mathbb{Z}^+$ and we could apply modular coloring. But is that possible?

Proposition 5 (Integralization). *Suppose $L = \{l_1, l_2, \ldots, l_n\} \subset \mathbb{R}_{>0}$. Then there exists $L' = \{l_1, l_2, \ldots, l_n\} \subset \mathbb{Z}_{>0}$ such that $\langle \pm LK_n \rangle$ and $\langle \pm L'K_n \rangle$ have the same biased graph, hence the same balanced chromatic polynomial.*