Gain Graphs and Hyperplane Arrangements
Lecture 18:
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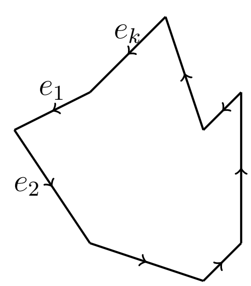
*Proof of Proposition* **??**. Let $L = \{l_1, \ldots, l_t\} \subset \mathbb{R}^+_{>0}$ where $\mathbb{R}^+_{>0}$ denotes the additive semigroup of positive real numbers. We will consider $\pm LK_n$, the gain graph with gain group $\mathbb{R}^+$, and the affinographic arrangement $\mathscr{A}[\pm LK_n]$. The objective is to find a set of integral gains $L' = \{l'_1, \ldots, l'_t\} \subset \mathbb{Z}^+_{>0}$ such that $\langle L'K_n\rangle = \langle LK_n\rangle$. To guarantee we obtain the same balanced chromatic polynomial, we must keep the same set of balanced circles. How do we achieve this? Consider a circle $C$ as in the figure below, where $\varphi(e_j) =$ some $\pm l_{i_j}$.




Here $\varphi(C) = \epsilon_1 l_{i_1} + \epsilon_2 l_{i_2} \cdots + \epsilon_k l_{i_k}$ where $\epsilon_j \in \{-1, 1\}$ and depends on $\varphi(e_j)$. If $C$ is balanced, then $\varphi(C) = 0$ and if $C$ is not balanced, then $\varphi(C) \neq 0$. We need to choose $L'$ so that this is preserves balance, i.e., $\varphi'(C) = 0$ if and only if $\varphi(C) = 0$. So we will have an equation or inequation of the form

$$\epsilon_1 l_{i_1} + \epsilon_2 l_{i_2} + \cdots + \epsilon_k l_{i_k} \begin{cases} = 0, \\ \neq 0. \end{cases}$$

That is almost enough to get the set $L$, We need $t$ distinct values. There is one simplification: since we use gains $\pm l_i$, it does not matter whether $l_i$ is positive or negative; but it must not be 0. Thus, we also need to state that $l_i \neq 0$ and $l_i \neq \pm l_j$ for $i \neq j$.

Now we replace the specific values $l_i$ with variables $x_i$ to obtain an equation $\epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_k l_{i_k} = 0$ or an inequation $\epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_k l_{i_k} \neq 0$. There are $t$ variables, $x_1, \ldots, x_t$. We have one equation per balanced circle, one inequation per unblaanced circle, and $t^2$ inequations $x_i \neq 0$, $x_i \neq x_j$, and $x_i \neq -x_j$ for $i \neq j$.

This gives us a system of equalities and inequalities. We look for an integer solution using linear algebra. That solution is guaranteed by Theorem 1. □

**Theorem 1** (Integralization)**.** *Given* $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \in \mathbb{Q}^t$*, the requirements that all* $\alpha_i \cdot x = 0$ *and all* $\beta_j \cdot x \neq 0$*, and the existence of a real solution* $y \in \mathbb{R}^t$*, then there exists a rational solution in* $\mathbb{Q}^t$*.*

After obtaining a rational solution, since $t$ is finite, we can scale by an appropriate integer to obtain an integral solution.

*Proof.* Let the matrix $A$ have rows $a_i^{\perp}$. Consider the equation $AX = 0 \in \mathbb{R}^p$. Any dot product $\alpha \cdot x = 0$ is forced if and only if $\alpha \in \mathrm{Row}(A)$. Therefore, none of the $\beta_j$'s are in $\mathrm{Row}(A)$. The solution space of $Ax = 0$ is $\mathrm{Nul}(A)$, which can be given in terms of parameters, say $x_1, \ldots, x_r$, $r \leq n$. $A$ is a $p \times t$ matrix. So we can put $A$ into RREF $\begin{bmatrix} I_k | A' \end{bmatrix}$ and the condition becomes $\begin{bmatrix} I_k | A' \end{bmatrix} \begin{pmatrix} \hat{x} \\ \bar{x} \end{pmatrix} = 0$ where $\hat{x} = (x_1, \ldots, x_k)$ and $\bar{x} = (x_{k+1}, \ldots, x_r)$. So $I\hat{x} + A'\bar{x} = 0$ and $\hat{x} = -A'\bar{x}$. Let $B = -A'$. Then

$$\mathrm{Nul}(A) = \left\{ \begin{pmatrix} I \\ B \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} : \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \in \mathbb{R}^r \right\} \subseteq \mathbb{R}^t.$$

So our solution is $\{ \begin{pmatrix} \bar{x} \\ B\bar{x} \end{pmatrix} : \bar{x} \in \mathbb{R}^t \}$. Because $A$ was a rational matrix and $B$ was obtained from $A$ by row operations, $B$ is also a rational matrix. Therefore $\bar{x} \in \mathbb{Q}^r$ implies $x \in \mathbb{Q}^t$. Therefore we have found a rational solution. We can choose $\bar{x}$ arbitrarily near $\bar{y}$ so that $x$ is arbitrarily near $y$. This means each $\beta_j \cdot x$ is changed too little to become 0, therefore we preserve all $\beta_j \cdot x \neq 0$. □

Now we know that we can replace real gains by rational gains and therefore by integral gains. This raises a natural question: How much (or little) do we need to perturb the real gains to obtain the rational gains? In particular, what is the smallest integer $d > 0$ such that perturbing $y$ by $< 1/d$ gives a rational solution? Put differently, what is the smallest $D$ such that rounding $Dy$ to the nearest integer vector gives a solution? More simply, we might try multiplying the real gains by $10^m$ (for some positive integer $m$) and then rounding to the nearest integer. But what $m$ is sufficiently large?

Virtually the same proof as that of Proposition **??** works for any additive real gain graph. Thus:

**Theorem 2** (Gain Graph Integralization)**.** *Let $\Phi$ be any $\mathbb{R}^+$-gain graph. Then there exists a $\mathbb{Z}^+$-gain graph $\Phi'$ with the same biased graph and therefore the same chromatic polynomials.*

We can infer even more extensive conclusions from Theorem 2. Every complex additive gain graph $\Phi$ can be replaced by a gain graph $\Phi'$ whose gains are Gaussian integers. Indeed, gains in any real vector space $\mathbb{R}^d$ can be replaced by vectors in $\mathbb{Z}^d$. The gains could even be polynomials over $\mathbb{R}$ or $\mathbb{C}$.

Notice that $\pm LK_n$ is similar to a hollow extended Catalan arrangement, which by definition is $\mathscr{A}[\pm[1,t]_{\mathbb{Z}}K_n]$ where $\pm[1,t]_{\mathbb{Z}}$ is the interval of integers from 1 to $t$. I will develop this thought in the final lecture.