Gain Graphs and Hyperplane Arrangements
Lecture 19:
Notetaker: Shuchen Mu

In [**?**, Section 5.5] Stanley introduces "generic" exponential sequences $\mathscr{A}[\pm LK_n]$, $n \geq 0$, where $L = \{l_1, \ldots, l_t\} \subset \mathbb{R}^+_{>0}$. He gives two proposed definitions of genericity.

(S1) $\mathcal{L}(\mathscr{A}[\pm LK_n])$ [that is, $\mathrm{Latb}(\pm LK_n)$] is as big as possible.

(S2) The $l_i$'s are linearly independent over $\mathbb{Q}$.

Does (S1) means $\pm LK_n$ has the fewest possible balanced circles? That suggesets another definition based on thinking about gain graphs:

(T1) $L$ is *generic* if, for all $n$, $\mathscr{B}(\pm LK_n)$ is as small as possible. Restated,

$$\mathscr{B}(\pm LK_n) = \bigcup_i \mathscr{B}(\pm l_i K_n),$$

since we necessarily have the balanced circles of $\pm K_n$.

(T1) is slightly different from (S2), since we only ask that no circle with different $l_i$'s in it can give gain 0 for any $n$, which is implied by rational independence of the $l_i$. However, it is easy to prove that because we have an exponential sequence they are equivalent.

**Proposition 1.** (T1) $\iff$ (S2).

*Proof.* Exercise. □

Now we examine the concept of a "bigger" semilattice or lattice of a biased graph.

**Proposal 2.** Given a graph $\Gamma$ and two linear classes of circles, $\mathscr{B}_1 \subset \mathscr{B}_2$, then $\mathrm{Latb}(\Gamma, \mathscr{B}_1)$ is bigger than $\mathrm{Latb}(\Gamma, \mathscr{B}_2)$.

"Bigger" is intentionally not defined, but it should mean something like existence of an order-preserving, or order- and rank-preserving, injective function $\mathrm{Latb}(\Gamma, \mathscr{B}_2) \to \mathrm{Latb}(\Gamma, \mathscr{B}_1)$ that is not surjective. Or, it may mean the existence of an order-preserving surjective function $\mathrm{Latb}(\Gamma, \mathscr{B}_1) \to \mathrm{Latb}(\Gamma, \mathscr{B}_2)$ that is not injective. The latter definition gives a proof of Proposal 2.

**Problem 3.** Decide whether both definitions of "bigger" are equivalent.

We get a better understanding of "bigger" from the following property of the balanced-flat semilattice.

**Theorem 4.** *Given a graph $\Gamma$ without loops and two linear classes of circles, $\mathscr{B}_1 \subset \mathscr{B}_2$, then there is an order-preserving surjective mapping* $\operatorname{Latb}(\Gamma, \mathscr{B}_1) \to \operatorname{Latb}(\Gamma, \mathscr{B}_2)$ *that is not injective.*

*Proof.* Write $\Omega_1 = (\Gamma, \mathscr{B}_1)$ and $\Omega_2 = (\Gamma, \mathscr{B}_2)$. A mapping $\beta : \operatorname{Lat}(\Omega_1) \to \operatorname{Lat}(\Omega_2)$ is defined by $\beta(S) = \operatorname{clos}_2(S)$, where $\operatorname{clos}_1$ is the closure in $\Omega_1$. (Notice that we define $\beta$ on all edge sets.) We have to prove that $\beta$ is order-preserving and surjective. It is obvious that it maps $\operatorname{Lat}\Omega_1 \to \operatorname{Lat}\Omega_2$. It maps balanced flats to balanced flats because, by the hypothesis, a balanced set of $\Omega_1$ is also balanced in $\Omega_2$.

Thus, the closure of $S$ in both biases is its balance-closure, defined by

$$\begin{aligned}\operatorname{bcl}_1(S) &= S \cup \{e \notin S : \exists\, C \in \mathscr{B}_1,\ C \cup e \in \mathscr{B}_1\} \\ &\subseteq S \cup \{e \notin S : \exists\, C \in \mathscr{B}_2,\ C \cup e \in \mathscr{B}_1\} = \operatorname{bcl}_2(S).\end{aligned}$$

It follows that $\operatorname{clos}_2(S) \supseteq \operatorname{clos}_1(S)$ for any balanced edge set.

Clearly, $\beta$ preserves set containment, that is, lattice order. We must prove $\beta$ is surjective. For $A \in \operatorname{Latb}\Omega_2$, choose an $\Omega_2$-basis $B$ of $A$. It is balanced and independent in $\Omega_2$, hence it is a forest, so it is balanced in $\Omega_1$. Thus, $\beta(B) = \operatorname{clos}_1(B)$ is balanced. Now,

$$B \subseteq \operatorname{clos}_1(B) \subseteq \operatorname{clos}_2(B) = A$$

so

$$A = \operatorname{clos}_2(B) \subseteq \operatorname{clos}_2(\operatorname{clos}_1(B) \subseteq \operatorname{clos}_2(A) = A.$$

Therefore, $\beta(\operatorname{clos}_1(B)) = A$. This proves $\beta$ is surjective from $\operatorname{Latb}\Omega_1$ to $\operatorname{Latb}\Omega_2$.

To prove $\beta : \operatorname{Latb}\Omega_1 \to \operatorname{Latb}\Omega_2$ is not injective, choose a circle $C \in \mathscr{B}_2 \setminus \mathscr{B}_1$. For $e \in C$, $e \notin \operatorname{clos}_1(C \setminus e)$ but $e \in \operatorname{clos}_2(C \setminus e)$ so $\operatorname{clos}_2(C \setminus e) = \operatorname{clos}_2(C)$. The same applies to another edge $f \in C$, which exists because there are no loops (one-edge circles). Now, $\operatorname{clos}_1(C \setminus e)$ and $\operatorname{clos}_1(C \setminus f)$ are two balanced flats in $\Omega_1$ with the same image, $\operatorname{clos}_2(C)$, under $\beta$. □

I believe the mapping $\beta$ is not always surjective from $\operatorname{Lat}\Omega_1$ to $\operatorname{Lat}\Omega_2$, but I leave that as an exercise.

**Corollary 5.** (S1) $\iff$ (T1).

*Proof.* Apply Theorem 4, since $\mathscr{L}(\mathscr{A}[\pm LK_n]) \cong \operatorname{Latb}(\pm LK_n)$. □

In other words, we have proved that Stanley's two definitions are equivalent. (I am ignoring the possibility that order-preserving injections give a different notion of bigness from surjections; cf. Problem 3.)

I purposely omitted proposing that $\mathrm{Lat}(\Gamma, \mathscr{B}_1)$ is bigger than $\mathrm{Lat}(\Gamma, \mathscr{B}_2)$. I suspect this is not always true.

**Problem 6.** Does there necessarily exist an order-preserving surjection $\mathrm{Lat}(\Gamma, \mathscr{B}_1) \to \mathrm{Lat}(\Gamma, \mathscr{B}_2)$?

**Biased union.** It is time to introduce a new way of combining biased graphs. The *biased union* of $\Omega_1 = (\Gamma_1, \mathscr{B}_1)$ and $\Omega_2 = (\Gamma_2, \mathscr{B}_2)$, whose edge sets are disjoint, is

$$\Omega_1 \sqcup \Omega_2 = (\Gamma_1 \cup \Gamma_2, \mathscr{B}_1 \cup \mathscr{B}_2).$$

**Theorem 7.** *The biased union is a biased graph.*

*Proof.* Exercise. □

The $n^{\text{th}}$ gain graph of our exponential sequence, $\pm LK_n$, contains $\pm l_i K_n$ for each $i$ (which is isomorphic to the hollow Catalan gain graph). Definition (T1) states that $L$ is generic if $\pm LK_n$ is the biased union

$$\langle \pm LK_n \rangle = \bigsqcup_{i=1}^{t} \langle \pm l_i K_n \rangle.$$

Now, here is the crucial question and the purpose of introducing the biased union. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Omega = \Omega_1 \sqcup \Omega_2$.

**Problem 8.** Can we express $\chi^b_{\Omega}(\lambda)$ in terms of $\chi^b_{\Omega_1}(\lambda)$ and $\chi^b_{\Omega_2}(\lambda)$ and possibly other information that we already know from $\Omega_1$ and $\Omega_2$?

Can we similarly infer $\mathrm{Latb}\,\Omega$?

Recall the formulas:

$$\chi^b(\lambda) = \sum_{\substack{S \subseteq E \\ \text{balanced}}} (-1)^{|S|} \lambda^{c(S)} = \sum_{A \in \mathrm{Latb}\,\Omega} \mu(\emptyset, A) \lambda^{c(A)}.$$

I would like to somehow use these formulas to extrapolate $\chi^b_\Omega$ from the sets that are balanced and closed in the two $\Omega_i$. For instance, (assuming no loops or balanced digons) common closed sets are $\emptyset$ and $\{e\}$ for every edge of the union. But then it gets complicated. For instance, every forest of the union $\Gamma$ is balanced, even if it combines edges of both $\Omega_i$, but not necessarily closed. Consider a subset $S \subseteq E(\Gamma)$: it is balanced and closed if and only if every block is balanced and closed. Suppose, then, that $S$ a block: it is balanced and closed if and only if it is a closed, balanced, inseparable subset in $\Omega_1$ or $\Omega_2$. So, a balanced flat of $\Omega$ is assembled from inseparable balanced flats of the $\Omega_i$. Does that give us enough insight to compute the balanced chromatic polynomial of $\Omega$, or even the semilattice of balanced flats?

In the special case of an exponential sequence $\langle \pm LK_n \rangle$ for generic $L$, perhaps some version of the exponential formula might be able to give a solution. That is the motivation for this discussion.

Here is a thought about generalization. It is surely too hard to solve in general, as even the special case of biased union is unclear.

**Problem 9.** Suppose we define

$$\Omega_1 \cup \Omega_2 = (V_1 \cup V_2, E_1 \cup E_2, \mathscr{B})$$

where $\mathscr{B}$ is the smallest linear class such that $\mathscr{B} \supseteq \mathscr{B}_1 \cup \mathscr{B}_2$. What is $\mathscr{B}$? What are the properties? Can we describe $\operatorname{Lat} \Omega_1 \cup \Omega_2$ or $\operatorname{Latb} \Omega_1 \cup \Omega_2$ in terms of $\Omega_1$ and $\Omega_2$?

The principal question here is whether $\mathscr{B}$ has an explicit description. Only then can any more be thought about.