GAIN GRAPHS AND HYPERPLANE ARRANGEMENTS
LECTURE 2
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Definition 1. Take a gain graph $\Phi = (\Gamma, \varphi)$, where φ takes values in $K^\times$, the multiplicative group of the field K. The hyperplane arrangement associated to the graph, $\mathcal{A}[\Phi]$, is in K^n, where n = number of vertices in the graph. It is $\mathcal{A}[\Phi] = \{h(e) : e \in E\}$, where

$$\begin{aligned} h(e_{ij}) &: x_j = x_i\varphi(e_{ij}) && \text{if } e_{ij} \text{ is the edge from } v_i \text{ to } v_j, \\ h(e_i) &: x_i = 0 && \text{if } e_i \text{ is a half edge with vertex } v_i, \\ h(e) &: 0 = 0 && \text{if } e \text{ is a loose edge.} \end{aligned}$$

For any edge set S, we define $h(S) := \{h(e) : e \in S\}$. Thus, $\bigcap h(S) := \bigcap_{e\in S} h(e)$.

Let's consider the gain of a circle. We defined the gain of a walk $W = e_{01}e_{12}\cdots e_{(l-1)l}$ as

$$\varphi(W) := \varphi(e_{01})\varphi(e_{12})\cdots\varphi(e_{(l-1)l}).$$

In a circle C, we choose one of the vertices as v_0 and start labeling all its vertices in one direction as $v_0, v_1, \ldots, v_l = v_0$. This defines the gain of C. The hyperplanes associated to the edges on the circle are:

$$\begin{aligned} h(e_{01}) &: x_1 = x_0\varphi(e_{01}), \\ h(e_{12}) &: x_2 = x_1\varphi(e_{12}), \\ &\cdots \\ h(e_{(l-1)0}) &: x_0 = x_{l-1}\varphi(e_{(l-1)0}). \end{aligned}$$

(For technical reasons we admit the whole space K^n as the "degenerate hyperplane".) The sequence of equations implies $x_0 = x_0\varphi(C)$.

Case 1. C is balanced, i.e., $\varphi(C) = 1$. If a point

$$x = (x_0, x_1, \ldots, x_n) \in \bigcap h(C \setminus \{e_{l-1,l}\}),$$

then $x_{l-1} = \varphi(e_{01})\varphi(e_{12})\cdots\varphi(e_{(l-2)(l-1)})x_0$, so $x_0 = \varphi(e_{(l-1)l})x_{l-1}$, so $x \in h(e_{(l-1)l})$. Therefore, $\bigcap_{i=1}^{l-1} h(e_{i-1,i}) \subseteq h(e_{l-1,l})$, so $h(C)$ is dependent.

Case 2. If C is unbalanced, then $x_0 = x_0\varphi(C)$ where $\varphi(C) \neq 1$, so $x_0 = 0$. It follows that $x_1 = 0, x_2 = 0, \ldots$, so

$$\bigcap h(C) = \{x \in K^n \mid x_i = 0 \ \forall\ v_i \in V(C)\}.$$

Between Cases 1 and 2 we have proved

Lemma 2 (Dependence of a Circle)**.** *If C is a balanced circle, then $h(C)$ is a dependent set of hyperplanes, but $h(C \backslash \{e\})$ is an independent set for every edge $e \in C$.*

If C is an unbalanced circle, then $h(C)$ is an independent set of hyperplanes.

Now we turn our attention to forests.

Lemma 3. *Let F be a forest in Φ. Then $h(F)$ is an independent set of hyperplanes.*

Proof. Here we think of a forest as an edge set.

Case 1. If there is only one edge in the forest, then it forms an independent singleton because its hyperplane is independent.

Case 2. Suppose all forests with a number of edges $\leq k$ have independent image under h. Let F be a forest that consists of $k+1$ edges.

Take a pendant edge e_{kl}, so there is no other edge than e_{kl} in the forest incident to v_l. Therefore the only defining equation of edges in the forest that involves x_l is $x_l = x_k\varphi(e_{kl})$. Let $F' = F \setminus \{e_{kl}\}$. Denote the defining vector of $h(e)$ by u_e. By the preceding discussion,

$$u_{e_{kl}} \notin \operatorname{span}\{u_e \mid e \in F'\},$$

so

$$\operatorname{rk}\operatorname{span}\{u_e \mid e \in F'\} < \operatorname{rk}\operatorname{span}\{u_e \mid e \in F\}.$$

By assumption, $h(F')$ is independent, so $\operatorname{rk}\operatorname{span}\{u_e \mid e \in F'\} = \#F'$. Hence

$$\operatorname{rk}\operatorname{span}\{u_e \mid e \in E(F)\} \geq \#F = \#F' + 1.$$

Thus, $h(F)$ is independent. □

Next, we need a little matroid theory.

Definition 4. The direct sum of two matroids, $M_1 \oplus M_2$, is defined by

$$E(M_1 \oplus M_2) = E(M_1) \sqcup E(M_2),$$
$$\operatorname{rk}_{M_1 \oplus M_2}(X) = \operatorname{rk}_{M_1}(X \cap E(M_1)) + \operatorname{rk}_{M_2}(X \cap E(M_2)).$$

Let's look at independence. Recalling that $\operatorname{rk}(X) \leq \#X$, we cna see that

$$\operatorname{rk}_{M_1}(X \cap E(M_1)) + \operatorname{rk}_{M_2}(X \cap E(M_2)) = \#X$$

if and only if

$$\operatorname{rk}_{M_1}(X \cap E(M_1)) = \#[X \cap E(M_1)],$$

and also

$$\operatorname{rk}_{M_1}(X \cap E(M_1)) = \#[X \cap E(M_1)].$$

That is, X is independent in $M_1 \oplus M_2$ if and only if its M_1 part and its M_2 part are independent in their respective matroids. This is a characteristic of the direct sum; if it is true for every X, then we have the direct sum of M_1 and M_2.

Consider how this applies to a hyperplane arrangement.

Lemma 5. *Let $\mathcal{B}, \mathcal{C}$ be arrangements in K^n, such that the coordinates of the hyperplanes in $\mathcal{B}$ are all different from those of the hyperplanes in $\mathcal{C}$. Then $\mathcal{M}(\mathcal{B} \cup \mathcal{C}) = \mathcal{M}(\mathcal{B}) \oplus \mathcal{M}(\mathcal{C})$.*

Proof. Since the coordinates of hyperplanes in $\mathcal{B}$ are different from those in $\mathcal{C}$, $\mathcal{B} \cup \mathcal{C} = \mathcal{B} \sqcup \mathcal{C}$.

Let X be an independent set in the matroid $\mathcal{M}(\mathcal{B} \cup \mathcal{C})$. There are no linear relations between defining vectors in the set $\mathcal{B}$ and those in the set $\mathcal{C}$. Thus, if $X \cap \mathcal{B}$ is independent in $\mathcal{M}(\mathcal{B})$ and $X \cap \mathcal{C}$ is independent in $\mathcal{M}(\mathcal{C})$, then X is independent in $\mathcal{M}(\mathcal{B}) \oplus \mathcal{M}(\mathcal{C})$, and conversely. Thus, $\mathcal{M}(\mathcal{B} \cup \mathcal{C}) = \mathcal{M}(\mathcal{B}) \oplus \mathcal{M}(\mathcal{C})$. □

Lemma 6. *Suppose $D \subseteq E(\Phi)$ is such that each component of D is a tree or contains only one circle, which is unbalanced, or a half edge. Then $h(D)$ is independent in $\mathcal{M}(\mathcal{A}[\Phi])$.*

Proof. let K be a component that contains one unbalanced circle C. Then by Lemma 2, $\bigcap_{e \in E(C)} h(e) = \{x : x_i = 0 \ \forall \ v_i \in V(C)\}$. Since K is connected, each vertex $v_j \in V(K)$ is linked to C by a path, say P from $v_i \in V(C)$ to v_j. So if $x \in \bigcap h(E(K))$, then $x_j = x_i \varphi(P) = 0$. Therefore, $x_j = 0$ for every $v_j \in V(K)$. On the other hand, if $x_j = 0$ for all $v_j \in V(K)$, then certainly $x_i \in \bigcap h(E(K))$. This shows that $\bigcap h(E(K)) = \{x : x_j = 0 \ \forall \ v_j \in V(K)\}$, which is of codimension $\#V(K)$, which $= \#E(K)$, so $E(K)$ is independent.

Let K be a component with a half edge attached to a vertex v_i, so $x_i = 0$. Since K is connected, by similar reasoning we conclude that $E(K)$ is independent.

By Lemma 3, if K is a tree, then $h(E(K))$ is independent.

As we saw in Lemma 5, this implies that $h(D)$ is independent. □

Lemma 6 begins to tell us how to define independent sets in the frame matroid of a $K^\times$-gain graph. In particular, if an edge set contains a balanced circle, then it is dependent.