Lecture 3 (Conclusion of Theorem 3)
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We finish the proof of Theorem 3. First, a few new definitions.

Definition 1. A *unicyclic graph* is a connected graph with exactly one circle. A 1-*tree* is a tree with or without a half edge. A *pseudotree* is a 1-tree or a unicyclic graph. A *pseudoforest* is a graph whose components are pseudotrees.

Definition 2. A gain graph is *contrabalanced* if every circle is unbalanced; i.e., it has no balanced circles.

Theorem 3. *Let Φ be a $K^{\times}$-gain graph, Let M be the matroid on $E(\Phi)$ that corresponds to $\mathscr{M}(\mathscr{A}([\Phi]))$. The independent sets in M are (the edge sets of) the contrabalanced pseudoforests in Φ.*

Recall that Lemma 3 says an edge set each of whose components is a tree, an unbalanced unicycle, or a tree with a single half edge is independent in the matroid M of the gain graph Φ that corresponds to the hyperplane arrangement $\mathscr{A}[\Phi]$ over a field K. I.e., any contrabalanced pseudoforest is independent in M.

So we must prove every independent edge set is a contrabalanced pseudoforest. We'll prove the contrapositive:

Lemma 4. *The dependent sets are the edge sets that have a single component with at least 2 circles, at least 2 half edges, or at least one circle and one half edge.*

Proof. Since a set containing a dependent set is dependent we only need to prove that a connected edge set that contains, two circles, two half edges, or a circle and half edge is dependent in M.

Suppose S is such an edge set. If S contains a balanced circle then we know it is dependent by Case 1 of the previous treatment of a circle. So we may assume every circle in S is unbalanced.

Case 1: S contains an unbalanced circle or half edge C_1, and one other one C_2, that share at most one vertex. There must be a minimal path P connecting C_1 and C_2 (see figure 2)

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By Case 2 of circles or by definition of a half edge hyperplane $h(e)$, C_1 forces $x_i = 0$ and C_2 forces $x_j = 0$.

Indeed, write $P = w_0 e_1 w_1 .... w_{l-1} e_l w_l$ so that in $\mathscr{A}[\Phi]$,

$$x_{w_l} = x_{w_{l-1}} \varphi_{l-1,l}(e_l) = ... = x_{w_0} \varphi(P).$$

But $x_{w_0} = 0$ from C_1, therefore $x_{w_l} = 0$.

We will show that some $e \in S$ has the property that

$$h(e) \supseteq \bigcap_{f \in S \setminus e} h(f).$$

(Every $e \in S$ has that property, but we don't need that.)

Pick e to be an edge in C_2 incident with v_j so $e = e_{jk}$ or the half edge e_j. With the equations $x_k = x_j \varphi_{jk}(e)$ for a circle and $x_j = 0$ a half edge.

Then we consider the equations separately.

In the half edge case $x_j = 0$ (consider C_1 and P) in $\bigcap h(C_1 \cup P) = \bigcap h(S \setminus e_j)$, therefore $h(S)$ is a dependent set of hyper planes.

In the circle case $x_j = 0$ and $x_k = 0$ (consider C_1 and the path $P \cup (C_2 \setminus e)$), then $x_k = x_j \varphi_{jk}(e)$ is satisfied (with $x_k = x_j = 0$) by $\bigcap h(S \setminus e)$; therefore $h(S)$ is a dependent set of hyperplanes.

This solves the problem when S contains two unbalanced circles/half edges with at most one vertex in common.

Case 2: Now we will consider the case where S contains two circles with at least two common vertices.

We treat first the case where S is a theta graph. We have 3 unbalanced circles, say $C_{12} = P_1 P_2^{-1}$, $C_{23} = P_1 P_3^{-1}$, $C_{13} = P_1 P_3^{-1}$ given by the internally disjoint paths of the theta graph, P_1, P_2, P_3, which all start and end at v_i and v_j, respectively. Then C_{12}, being unbalanced, implies $x_i = x_j = 0$. Let $e = e_{jk}$ be the edge in p_3 at v_j Then $C_{12} \cup (P_3 \setminus e)$ implies $x_k = 0$ thus $x_k = 0$ in $\bigcap h(C_{12} \cup (P_3 \setminus e)) = \bigcap h(S \setminus e)$. Similarly, $x_j = 0$. Therefore $h(e) \supseteq h(S \setminus e)$, so the theta graph is dependent in M. So if S contains a theta graph, it is dependent.

Note that we have been assuming S is connected. Let $\xi(S)$ be the *cyclomatic number* of S, defined as the smallest number of edges that when deleted leave a tree spanning $V(S)$, i.e., the minimum number of edges whose deletion leave a forest (therefore a tree). We have $\xi(S) > 1$ because C_1 and C_2 exist. If $\xi(S) = 2$ then S contains the theta graph $C_1 \cup C_2$ (because those two circles have at least two common vertices), so we are done. If $\xi(S) > 2$ then we can delete $\xi(S) - 2$ edges to get a connected subgraph S' with $\xi(S') = 2$, which contains a theta graph or handcuff (by easy graph theory) and is therefore dependent.

So we have finished the proof modulo some graph-theoretic detail. □