Gain Graphs and Hyperplane Arrangements
Lecture 4 (The Frame Matroid)
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In the last lecture, we analyzed the matroid $\mathscr{M}(\mathscr{A}[\Phi])$ in terms of the independent sets of Φ. Theorem 1 showed that the independent sets are contrabalanced pseudoforests.

Definition 1. Let $\mathbf{F}(\Phi)$ be this matroid on a $K^\times$-gain graph Φ, with independent sets given by Theorem 1. This is the *frame matroid* of Φ.

Theorem 1. *For the gain graph Φ, let $S \subseteq E(\Phi)$. Then,*

(1) $rk(S) = n - b(S)$.

(2) *The circuits of $F(\Phi)$ belong to one of the following three categories:*

  (a) *Balanced circles and loose edges.*

  (b) *Contrabalanced handcuffs.*

  (c) *Contrabalanced theta graphs.*

(3) *The closure of S is given by*

$$cl(S) = (E{:}V_0(S)) \cup bcl(S{:}V_0(S)^c.$$

(4) *S is closed if S is equal to its closure (i.e. the union of unbalanced components is an induced subgraph of Φ and each balanced component is balance-closed)*

To explain the notation we give some additional definitions.

Definition 2. For $S \subseteq E$, we define $b(S)$ to be the number of balanced components of the spanning subgraph (V, S). Isolated vertices are included as balanced components because they are balanced. However, loose edges are not included as they are not considered components.

We define $V_0(S)$ to be the set of vertices that are contained in unbalanced components of S.

Definition 3. For $X \subseteq V$ and $S \subseteq E$, the *induced subset of S on X*, denoted by $S{:}X$, is defined as the set $\{e \in S : \text{all endpoints of } e \text{ are in } X\}$. Loose edges are included in the induced subset, as they do not contain a vertex that is not in X.

Definition 4. The *balance-closure of S*, denoted by $\mathrm{bcl}(S)$, is given by

$$S \cup \{e \in E : \exists \text{ balanced circle } C \text{ such that } e \in C \subseteq S \cup \{e\}\}$$
$$\cup \{\text{loose edges}\},$$

which is equal to

$$\begin{aligned} S \cup \{e \in E : \exists \text{ path } P \text{ in } S \text{ joining the endpoints of } e \\ \text{such that } P \cup \{e\} \text{ is a balanced circle}\} \\ \cup \{\text{balanced loops}\} \cup \{\text{loose edges}\}. \end{aligned}$$

In both sets the loose edges could be omitted if we consider loose edges to be balanced circles.

An edge set S is *balance-closed* if $S = \text{bcl}(S)$.

Before we begin the proof of the theorem, there are two notes about the balance-closure of S. First, we do not yet know if the balance-closure satisfies all three conditions to be an abstract closure. Clearly, $S \subseteq \text{bcl}(S)$ and balance-closure is weakly increasing (i.e., adding edges to S cannot remove an edge from $\text{bal}(S)$). However, it may not be true that $\text{bcl}(\text{bcl}(S)) = \text{bcl}(S)$. Secondly, the balance-closure of a balanced set must be balanced, because it contains no unbalanced circles or half-edges. This requires proof.

Lemma 1. *If S is balanced, then bcl(S) is balanced.*

Proof. Switch so S has all identity gain. Then an edge added to S by balance-closure must also have identity gain. It follows that $\text{bcl}(S)$ is balanced. □

We begin the proof of the theorem with part (2).

Proof of Part (2). Let D belong to one of the categories described in this part and let $e \in D$. From Theorem 1, we know that D is dependent and that $D \setminus e$ is independent. Independence comes from the fact that if we were to remove any edge from D, this would result in a contrabalanced pseudoforest. Therefore D is a circuit.

Conversely, let D be a circuit. Then D is dependent. By Theorem 1, D must contain a balanced circle or two unbalanced circles/half-edges that are in the same component of D (since if they were in different components, they would not make D dependent). In Theorem 1 we already showed that this subset D' of D is dependent and is either a balanced circle or a contrabalanced handcuff or theta graph. By minimality, $D = D'$. □

Proof of Part (1). From the definition of $\mathbf{F}(\Phi)$ (as corresponding to $\mathscr{M}(\mathscr{A}[\phi])$) we have $\text{rk}(S) = \text{rk}(\bigcap h(S))$, as the right-hand side is the rank function in the hyperplane arrangement. This is then equal to $\text{codim} \bigcap h(S)$. We know that $\text{rk}(S) = \text{rk}(I)$ for any maximal contrabalanced pseudoforest I because it is a maximal independent set. Thus we also have $\text{rk}(S) = \text{rk}(I) = \text{codim} \bigcap h(I)$.

The following is a diagram of the structure of I.

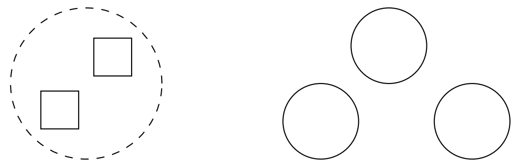

FIGURE 1. Structure of I. Each box represents an unbalanced component R_i of I and each circle represents a balanced component I_i of I. The dashed circle represents $V_0(S)$.

We can break up I into its unbalanced components and its balanced ones. Collectively, the unbalanced components are $I{:}V_0(S)$. For the balanced components, each component must be a tree because I is independent. (We are able to ignore loose edges as they correspond to the degenerate hyperplane.)

For an unbalanced component R_i, we found in the study of circles and the proof of Theorem 1 that $h(R_i) \implies x_j = 0$ for all $v_j \in V(R_i)$. Hence, we have a contrabalanced 1-tree. Therefore codim $\bigcap h(R_i)$ will be equal to the number of coordinates of the hyperplane equations in $h(R_i)$. However, this is just the number of vertices of R_i.Therefore $\mathrm{rk}(R_i) = |V(R_i)|$.

Next, we consider a balanced component I_i. It is a tree, by Theorem 1. All coordinates of $\bigcap h(I_i)$ are determined through the equations of the tree by fixing any one coordinate arbitrarily (see Figure 2). Therefore codim $\bigcap h(I_i) \geq |V(I_i)| - 1 = |E(I_i)$.

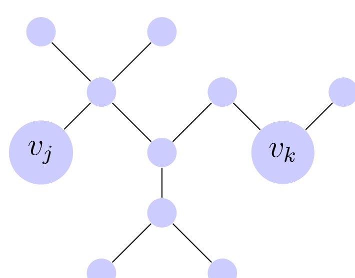

FIGURE 2. A possible structure for I_i. By fixing x_j, we can find x_k in terms of x_j.

Since codim $\bigcap h(I_i) \leq |E(I_i)|$ because the codimension cannot be greater than the number of hyperplanes of $h(I_i)$, therefore codim $h(I_i) = |V(I_i)| - 1$.

Now, since each $h(R_i)$ and each $h(I_i)$ uses a different set of coordinates, we can write $\mathrm{rk}(I)$ as $\sum_i \mathrm{rk}(R_i) + \sum_i \mathrm{rk}(I_i)$ (see the previous lemma about direct sums). Using the formulas we have just constructed, this becomes $\sum_i |V(R_i)| + \sum_i (|V(I_i)| - 1) = |V(I)| - b(I)$. Each vertex is included in either some R_i or some I_i, so this is $n - b(I)$.

For the proof of Part (1) to be complete, it remains to be shown that the vertex sets of the balanced components of I are those of the balanced components of S; in other words, that I restricted to a balanced component of S is connected. □

Lemma 2. *Suppose I is a maximal independent set in $S \subseteq E$. Let S have unbalanced components $U_1, U_2, \ldots$ and balanced components $B_1, B_2, \ldots$. Then $I \cap B_i$ is a tree spanning $V(B_i)$ and $I \cap U_i$ is a disjoint union of contrabalanced 1-trees.*

Proof. To appear next time. □