Gain Graphs and Hyperplane Arrangements
Lecture 5
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In the last lecture we ended with the statement of the following lemma which is necessary to complete the proof of Lemma 1 of Lecture 4. We begin with proving this lemma.

Lemma 1. Let Φ be a $K^{\times}$-gain graph, $S \subseteq E(\Phi)$, and I a maximal independent set contained in S. Then $b(I) = b(S)$.

Proof. Suppose S has balanced components $B_1, B_2, \ldots$ and unbalanced components $U_1, U_2, \ldots$. To prove that I and S have the same number of balanced components we will show two things:

(1) $I \cap E(B_j)$ is a spanning tree of B_j (and therefore balanced),
(2) $I \cap E(U_j)$ is a contrabalanced spanning 1-forest (and therefore has no balanced components).

For (1), notice that $I \cap B_j$ is a forest. This is because B_j is balanced, $I \subseteq B$, and I is independent. (If $I \cap B_j$ were not a forest there would be a balanced circle contained in I, but we know balanced circles are dependent. So $I \cap B_j$ is a forest.) Suppose it is disconnected. Then we can use an edge of B_j to join two trees of $I \cap B_j$ into one tree, since B_j is connected. Therefore, $I \cap B_j$ was not maximal, and thus I was not maximal. But we chose I to be maximal, so this is a contradiction. Thus $I \cap B_j$ is a tree, which spans B_j because an isolated vertex is a (very small) tree, so it falls under the previous argument.

For (2), notice that the only possible balanced components of $I \cap U_j$ are trees. By way of contradiction, suppose $I \cap U_j$ has a tree component, T. If T is not the only component of $I \cap U_j$ then we can add an edge of U_j to I in order to connect T to another component U of $I \cap U_j$ (this other component of $I \cap U_j$ will necessarily be either a tree, a tree with a single half-edge, or an unbalanced unicycle because I is a contrabalanced pseudoforest), giving a larger independent set (because connecting T to a component which is either a tree, a tree with a single half-edge, or an unbalanced unicycle$U$ will just yield a larger component of the same type as U, preserving the contrabalanced pseudoforest). Since we chose I to be maximally independent, this is a contradiction. So T must be the only component of $I \cap U_j$. Then either U_j has a half-edge e, in which case $T \cup e$ is a larger independent set, or U_j has an unbalanced circle C, which can be extended to a unicycle U that spans U_j (since U_j is connected). We can replace T by U to create an independent set larger than I. This is a contradiction because maximal independent sets all have the same size (by matroid theory). So $I \cap U_j$ has no tree components. Therefore $I \cap U_j$ has no balanced components. So in I, we get exactly one balanced component for each B_j and no others. Therefore, $b(I) = b(S)$. □

N.B. Once we have established the independent sets of the frame matroid $\mathbf{F}(\Phi)$ using $\mathscr{M}(\mathscr{A}[\Phi])$, we can define $\mathrm{rk}(S) = \max\{\#I \mid I \subseteq S,\ I \text{ independent}\}$ for $S \subseteq E(\Phi)$ entirely in terms of the gain graph. We never need to refer back to $\mathscr{A}[\Phi]$. This permits a vast generalization. Take any biased graph $(\Gamma, \mathscr{B})$, possibly from a gain graph and possibly not, and define $S \subseteq E$ to be independent if it is a contrabalanced pseudoforest.

Theorem 2. *Define* $\mathbf{F}(\Gamma, \mathscr{B}) := ((\Gamma, \mathscr{B}), \mathscr{I})$ *where* $\mathscr{I}$ *is the set of all* $S \subseteq E(\Gamma)$ *which induce contrabalanced pseudoforests. Then* $\mathbf{F}(\Gamma, \mathscr{B})$ *is a matroid on* E.

Proof. This proof will be omitted or postponed. □

Given Theorem 2, our proof that $\mathrm{rk}(S) = n - b(S)$ for $S \subseteq E$ and our proof of the circuits of $\mathbf{F}(\Phi)$ both apply to $\mathbf{F}(\Gamma, \mathscr{B})$.

The next step in the proof of Theorem **??** is to establish the closure operator.

Lemma 3. $\mathrm{cl}_{\mathbf{F}}(S) = (E{:}V_0(S)) \cup \mathrm{bcl}(S{:}V_0(S)^c)$.

For this we will establish a helpful lemma.

Lemma 4. Let Φ be a gain graph and let $R \subseteq E$ be balanced and connected. Then $\mathrm{bcl}(R)$ (aside from the balanced loops and loose edges) is the maximal balanced subset of E that contains R and has vertex set contained in $V(R)$.

Proof. Since R is balanced we can switch Φ so it has all identity gain. Then $\mathrm{bcl}(R) = \{e \in E{:}V(R) \mid \varphi(e) = \epsilon\}$ together with all balanced loops and loose edges. □

For an arbitrary edge set R we have a reduction to components, which is of most use when R is balanced.

Lemma 5. Let Φ be a gain graph and $R \subseteq E$. Let Ω be the set of components of R. Then $\mathrm{bcl}(R) = \bigcup_{C \in \Omega} \mathrm{bcl}(C)$.

Proof. An exercise. □

Balance-closure differs from closure in that bcl never joins unbalanced components of R, but cl can join them.