Lecture 6 (Closure and Closed Sets)
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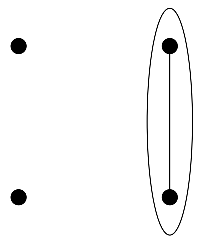
Let's recall some definitions from graph theory. Let $\Gamma = (V, E)$ be a graph and $S \subseteq E$. (We always use V and E for the edge sets of the graph Γ.) The vertex set of S is $V(S) := \{v \in V \mid \exists\, e \in S \text{ such that } v \text{ is an endpoint of } e\}$. Then for any $X \subseteq V$ satisfying $X \supseteq V(S)$, (X, S) is also a graph. (This is how every subgraph of Γ is formed.) For example, in the graph Γ below, V consists of all four vertices. If S is the singleton whose element is the edge, then the elements of $V(S)$ are the two vertices within the ellipse.



Moving over to a gain graph (or biased graph), V and E always denote its vertex and edge set.

As for "spanning", we are using it for two incompatible notions. (Not our fault; it's "Tradition!".) In graph theory, a *spanning subgraph* of a graph Γ is a subgraph whose vertex set is all of V. In matroid theory, a *spanning set* of a matroid is a subset of the ground set whose closure is the entire ground set. When an edge set S is said to be spanning, it is spanning in the matroid sense, because we are working with matroids on the edge set (and because an edge set is not a subgraph).

Finally, let's recall a definition from matroid theory. Let M be a matroid with ground set E. The *closure* of $S \subseteq E$ is $\mathrm{cl}(S) := S \cup \{e \in E \mid \mathrm{rk}(S \cup \{e\}) = \mathrm{rk}(S)\}$. We will be proving a more graphic formula in the next theorem. *N.B.* By matroid theory, since loose edges and balanced loops have rank 0, they are in every closed edge set.

Additional notation: We denote the number of balanced components of any subgraph (X, S) of a gain or biased graph, where $X \subseteq V$ and $S \subseteq E$ (necessarily with $V(S) \subseteq X$), by $b(X, S)$. Thus, for instance, our usual notation $b(S)$ is shorthand for $b(V, S)$.

Theorem 1 (Theorem **??**(3))**.** *Let $\Phi = (\Gamma, \varphi)$ be a gain graph with edge set E and $S \subseteq E$. Then*

$$\mathrm{cl}(S) = (E{:}V_0(S)) \cup \mathrm{bcl}(S{:}V_0(S)^c).$$

(All the loose edges and balanced loops are included in $\mathrm{bcl}(S{:}V_0(S)^c)$.) This follows from a lemma:

Lemma 2. *The balance-closure*[1] *of a balanced edge set B is*

$$\mathrm{bcl}(B) = B \cup \{e \in E{:}V(B) \mid B \cup \{e\} \text{ is balanced}\} \cup \{\text{loose edges and balanced loops}\}$$

Proof. Recall that $\mathrm{bcl}(B) = B \cup \{e \in E(\Gamma) \mid \exists \text{ a path } P \text{ in } B \text{ joining endpoints of } e \text{ such that } P \cup \{e\} \text{ is balanced}\} \cup \{\text{loose edges}\}$. Let A be the set in the statement of the lemma. As $\mathrm{bcl}(B)$ is balanced by Lemma **??**, $\mathrm{bcl}(B) \subseteq A$.

[1]Reiminder: Not "balanced closure".

Conversely, let $e \in (E{:}V(B)) \setminus B$. Then the endpoints of e are joined by a path P in B. But $B \cup \{e\}$ is balanced, as $e \in A$, so $P \cup \{e\}$ is a balanced circle, and $e \in \operatorname{bcl}(B)$. □

Proof of Theorem 1. Recall that $\operatorname{rk}(S) = n - b(S)$, so $\operatorname{cl}(S) = S \cup \{e \in E \mid b(S \cup \{e\}) = b(S)\}$. Clearly, no edge of $E : V_0(S)$ increases $b(S)$ when added to S: the balanced components remain the same. It's also clear that $\operatorname{cl}(S)$ can't contain any edge that connects a balanced component of S to any other component, as that would reduce $b(S)$.

The only remaining case is an edge whose endpoints are in a balanced component B. Again by Lemma (?), $\operatorname{bcl}(B)$ is balanced, so replacing B by $\operatorname{bcl}(B)$ preserves $b(S)$ and makes S larger (or leaves it unchanged).

Suppose we add an edge $e \in (E : V(B)) \setminus \operatorname{bcl}(B)$. By the above lemma, $B \cup \{e\}$ is not balanced. This reduces $b(S)$ by 1, so $e \notin \operatorname{cl}(S)$. □

Theorem 3 (Theorem **??**(4))**.** *A set $S \subseteq E$ is closed if and only if*

$$S = (E{:}V_0(S)) \cup \operatorname{bcl}(S{:}V_0(S)^c) \cup \{\textit{loose edges and balanced loops}\}.$$

Proof. By definition, S is closed iff $S = \operatorname{cl}(S)$. □

The preceding description is equivalent to the following: S is closed iff its unbalanced part is an induced edge set, each balanced component is balance-closed, and all loose edges and balanced loops of Φ are in S. We get all closed sets by taking all sets of the form

$$(E{:}X) \cup B \cup \{\text{loose edges and balanced loops}\}$$

where $X \subseteq V$, $E{:}X$ has no balanced components, and $B \subseteq E{:}X^c$ is balanced and balance-closed.

Graphs vs. gain graphs vs. biased graphs. The notion of biased graph is not exactly a generalization of that of a gain graph. Rather, it is an *abstraction* of a gain graph, because a biased graph does not have any gains; it retains only the combinatorial structure of a gain graph.

Similarly, a gain graph is not a generalization of a graph but a graph *with additional structure*. While it is true that any graph Γ can be made a $\mathfrak{G}$-gain graph for any group $\mathfrak{G}$ by declaring $\varphi(e) = \varepsilon$ (the identity) for all $e \in E(\Gamma)$, this is not always the "right" choice. It is "right" in the following sense: the hyperplane arrangement of Γ without a gain is exactly the same as the arrangement of Γ with all-identity gains. However, for a signed graph, which is a Z_2-gain graph, in some generalizations of ordinary graph theory (e.g., in regard to line graphs) it is preferable to assign every edge a negative sign, i.e., gain -1 rather than 1.

Gains vs. weights; orientation vs. direction. Let's clarify two subtle distinctions.

A *gain* is not the same as a *weight*. A gain is inverted when using the opposite orientation, while a weight is not (there need not even be orientation). (This is my personal distinction. It seems to be consistent with general usage, although it is not rigorously followed.)

The difference between an orientation, for the purpose of defining the gain, and no orientation shows up clearly in directed graphs. In a directed graph with gains, or *gain digraph*, each edge has only one direction. For example, a path or any walk must follow the directions of the edges; therefore, not every circle can be said to have a gain, but only the ones that are consistently directed.

Preview: The chromatic polynomial of a gain or biased graph

Recall that for a graph $\Gamma = (V, E)$, the *chromatic polynomial of* Γ is

$$\chi_\Gamma(\lambda) = \sum_{S \subseteq E} (-1)^{\#S} \lambda^{c(S)} = \sum_{A \in \mathrm{Lat}(E)} \mu(\emptyset, A) \lambda^{c(A)},$$

where $c(S) = c(V, S)$ is the number of components of (V, S). The definition in a biased graph, includings a gain graph, is very similar; there is only one little difference.

Definition 4. For a biased graph $\Omega = (\Gamma, \mathscr{B})$, where E is the edge set of Γ, define

$$\chi_\Omega(\lambda) := \sum_{S \subseteq E} (-1)^{\#S} \lambda^{b(S)} = \sum_{A \in \mathrm{Lat}(\Omega)} \mu(\varnothing, A) \lambda^{b(A)}.$$

The second equality follows from the Möbius function formula for matroids mentioned in Stanley's notes.

This polynomial agrees with one we already know from Stanley's notes.

Theorem 5. *For a $K^\times$-gain graph Φ, $p_{\mathscr{A}[\Phi[}(\lambda) = \chi_\Phi(\lambda)$.*

Proof. This follows from the rank formula $\mathrm{rk}(S) = n - b(S)$ and the known formulas for the characteristic polynomial of a hyperplane arrangement. □