Lecture 7: The Chromatic Polynomial
Nov. 20, 2019
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Recall that for a graph $\Gamma = (V, E)$, the chromatic polynomial of Γ is

$$\chi_\Gamma(\lambda) = \sum_{S \subseteq E} (-1)^{\#S} \lambda^{c(S)} = \sum_{A \in \operatorname{Lat}\Gamma} \mu(\varnothing, A)\lambda^{c(A)},$$

where $c(S) = c(V, S)$ is the number of components of (V, S) and $\operatorname{Lat}\Gamma$ is the lattice of flats of the graphic matroid of Γ. The definition in a biased graph, including a gain graph, is very similar; there is only one big little difference.

Definition 1. The *chromatic polynomial* of a biased graph $\Omega = (\Gamma, \mathscr{B})$, where E is the edge set of Γ, is

$$\chi_\Omega(\lambda) := \sum_{S \subseteq E} (-1)^{\#S} \lambda^{b(S)} = \sum_{A \in \operatorname{Lat}\Omega} \mu(\varnothing, A)\lambda^{b(A)}.$$

The second equality follows from the Möbius function formula for matroids mentioned in Stanley's notes.

If S is balanced then $b(S) = c(S)$, so if Ω is balanced the chromatic polynomial of the gain graph agrees with the chromatic polynomial of the underlying graph.

Recall our definition that, if the empty set is not closed, then $\mu(\varnothing, A) = 0$ for every flat A. Consequently, if Ω contains a loose edge or a balanced loop, then its chromatic polynomial is identically 0.

Definition 2. A biased graph has a second chromatic polynomial. First we have to define the *meet semilattice of balanced flats*,

$$\operatorname{Lat}^{\mathrm{b}}(\Omega) := \{A \in \operatorname{Lat}\Omega : A \text{ is balanced}\}.$$

The *balanced chromatic polynomial* of Ω is

$$\chi^{\mathrm{b}}_\Omega(\lambda) := \sum_{S \subseteq E:\text{balanced}} (-1)^{\#S} \lambda^{b(S)} = \sum_{A \in \operatorname{Lat}^{\mathrm{b}}\Omega} \mu(\varnothing, A)\lambda^{b(A)}.$$

If Ω is balanced, the balanced chromatic polynomial, like the chromatic polynomial, is the same as the chromatic polynomial of Γ. In other words, all three coincide. However, if Ω is not balanced, all three are different. (I will not prove that, but you can see that the range of summation for χ^{b}_Ω is smaller than that for χ_Ω and χ_Γ, and the exponents in the two latter sums are unequal for unbalanced sets S.

We now come to a nice generalization of the theorem about the characteristic polynomial of a graphic hyperplane arrangement (in Stanley's notes).

Theorem 3. *For a $K^\times$-gain graph Φ, $p_{\mathscr{A}[\Phi[}(\lambda) = \chi_\Phi(\lambda)$.*

Proof. In $p_{\mathscr{A}[\Phi[}(\lambda)$, the exponent $\dim h(S) = n - \operatorname{rk} h(S) = b(S)$ because $\operatorname{rk} h(S) = \operatorname{rk}_\Omega(S) = n - b(S)$. (Recall that for $S \subseteq E$, $h(S)$ is the set of corresponding hyperplanes and that the frame matroid $\mathbf{F}(\Phi)$ is isomorphic by h to $\mathscr{M}(\mathscr{A}[\Phi])$.) $\square$

Now that we have a polynomial defined on a gain graph. let's see what it can do.

Definition 4. Given a gain graph Φ we define a *k-coloration* as a function

$$\gamma : V(\Phi) \to \mathfrak{G} \times [k] \cup \{0\}.$$

A *zero-free k-coloration* is a coloration that does not use 0; in other words, it is a function

$$\gamma : V(\Phi) \to \mathfrak{G} \times [k].$$

We call the codomain the *color set* and denote it by $\mathbf{C}_k^0(\mathfrak{G})$, or $\mathbf{C}_k(\mathfrak{G})$ when we do not include 0.

Define a right action of $\mathfrak{G}$ on $\mathbf{C}_k^0$ and thus on $\mathbf{C}_k$ by $0g := 0$ and $(h, i)g := (hg, i)$ for $g, h \in \mathfrak{G}$ and $i \in [k]$. A coloration is *proper* if, for every ordinary edge e_{vw}, $\gamma(w) \neq \gamma(v)\varphi(e_{v,w})$, and also $\gamma(v) \neq 0$ for each vertex v that supports a half edge or an unbalanced loop.

We come at last to the main result of today's lecture.

Theorem 5 (Proper Coloration)**.** *If $\mathfrak{G}$ is finite, say of order m, then $\chi_\Phi(km+1)$ is the number of proper k-colorations of Φ and $\chi_\Phi^b(km)$ is the number of zero-free proper k-colorations.*

Observe the interesting fact that the number of proper colorations is independent of the particular group. It depends only on how big the group is.

Example 6 (Gain Graphs vs. Hyperplane Arrangements)**.** Let $\mathfrak{G} = K^\times$ and $k = 1$. Then $\mathbf{C}_1 = K$ (as a multiplicative monoid) and $\mathbf{C}_1^0 = K^\times$.

We can generalize this so as to be able to color the gain graph of a gain-graphic hyperplane arrangement. Let $\mathfrak{G}$ be any finite subgroup of $K^\times$. Examples are a finite cyclic group of any order as a subgroup of $\mathbb{C}^\times$ and the multiplicative group of a finite field $\mathbb{F}_q$ for cyclic groups of order $q - 1$ (q being a prime power).

Example 7 (Signed Graphs)**.** The special case of a group of order 2 is exceptionally interesting. Let $\mathfrak{G} = \{\pm 1\} \subseteq K^\times$, where $\operatorname{char} K \neq 2$ (that is, $1 \neq -1$). We call such a gain graph Φ a *signed graph*. We can treat the color set as a sign-symmetric set of integers if we color with $\{0, \pm 1, \pm 2, \ldots, \pm k\}$—as long as $\operatorname{char} K$ is large enough (the colors must be distinct), and in particular whenever $\operatorname{char} K = 0$.

In signed graphs, $\chi_\Phi(2k + 1)$ gives us the number of proper k-colorations when λ is odd, provided we set $k = \frac{1}{2}(\lambda - 1)$; and $\chi_\Phi^b(2k)$ gives us the number of zero-free proper k-colorations when λ even, if we take $k = \frac{1}{2}\lambda$.

There is extensive literature on signed graphs, though not much on their coloring. Much of it is not in mathematics, but is directed towards social science, inspired by a foundational article of Cartwright and Harary from 1956. On the other hand, the hyperplane arrangements of signed graphs are implicitly connected with the major mathematical topic of Lie theory via root systems (*q.v.*), which are becoming important in combinatorial geometry.

Definition 8. Sometimes we only want a 0-free 1-coloration; we call that a *group coloration* as it simply maps $V \to \mathfrak{G}$ (notationally simplified from $\mathfrak{G} \times [1]$).

Sometimes we like to regard the set $[k]$ as the group $\mathbb{Z}_k$ and view a k-coloration as a group coloration, just as we can when $k = 1$.