Lecture 8: Deletion, Contraction, and Coloration

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The main topic of this lecture is coloring gain graphs, but for the principal theorems we have to define not only deletion of edges, which is obvious, but also contraction of edges, which is far from obvious for gain graphs, in contrast to how it is for ordinary graphs.

Deletion and Contraction. We begin with gain graphs, then apply our ideas to biased graphs.

Definition 1 (Deletion of an Edge)**.** Deletion of an edge in a gain graph or a biased graph is trivial. It should be noted that all gains remain the same upon deletion and the balanced circles remain the same, except for those that are no longer circles upon the deletion.

Definition 2 (Contraction of an Edge in a Gain Graph)**.** The notation for a gain graph Φ with e contracted is Φ/e.

To contract a link e in Φ, first switch Φ so e has gain ϵ, then coalesce the endpoints, and finally delete the contracted edge e.

To contract a loose edge or a balanced loop, simply delete the edge. Do not change the gains of other edges.

To contract a half edge or an unbalanced loop e incident with vertex v, remove both v and e but not any other edges. This may remove some endpoints of some edges; in particular, it reduces a link e that joins v to w to a half edge incident with w, and a loop or half edge at v (other than e itself) to a loose edge. Do not change the gains of edges that remain ordinary edges—but an ordinary edge that becomes a half or loose edge no longer has a gain.

Many different switching functions can give e the switched gain ϵ, so the contraction Φ/e is not uniquely defined. All different contractions are switching equivalent, but the explanation is somewhat complicated so we postpone it to a later date.

Since a biased graph has no gains, the definition of contraction has to be adapted, and in such a way that it is compatible with contraction in a gain graph.

Definition 3 (Contraction of an Edge in a Biased Graph)**.** For a biased graph $\Omega = (\Gamma, \mathscr{B})$, again there are different rules for different kinds of edge. The notation for Ω with e contracted is Ω/e.

To contract a link e, we contract it in the underlying graph Γ. Then we have to define the bias. A circle D in Ω/e is balanced if it is the contraction of a balanced circle C in Ω that contains e, or if it is a circle in Ω that is balanced. Otherwise, it is unbalanced; that is, if it is the contraction of anun balanced circle in Ω that contains e, or if it is an unbalanced circle in Ω.

To contract a loose edge or a balanced loop, simply delete the edge.

To contract a half edge or an unbalanced loop e incident with vertex v, remove both v and e but not any other edges. This may remove some endpoints of some edges; in particular, it reduces a link e that joins v to w to a half edge incident with w, and a loop or half edge at v (other than e itself) to a loose edge.

It is worthwhile to point out why this is a complete definition of the balanced circle class in Ω/e. Suppose we contract a link e. A circle C in Ω that contains e will contract to a circle with edge set $C \setminus e$ in Γ/e, and whether or not it is balanced will not be affected by contraction. If C does not contain e, there are two cases. If both endpoints of e, say v and w, are in C, then C contracts to a pair of circles, each consisting of one of the two vw-paths in C. Otherwise, C simply remains a circle in the contracted graph.

There is one little difficulty. Unlike with gain graphs, where it is obvious, how do we know the contracted biased graph is a biased graph?

Proposition 1. *If Ω is a biased graph and e is an edge of Ω, then Ω/e is a biased graph.*

Proof. This is an excellent exercise for the reader. □

Given a gain graph Φ, we denote the corresponding biased graph by $\langle\Phi\rangle$. It should be clear that, if we have a gain graph Φ and contract an edge e, then take the biased graph of Φ/e, we get the same result as if we contract e after taking the biased graph of Φ. Symbolically,

$$\langle\Phi/e\rangle = \langle\Phi\rangle/e.$$

Example 1. We do an example of deletion and contraction using the $R^\times$-gain graph Φ in Figure 1.

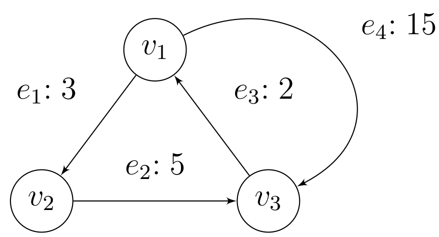

FIGURE 1. A gain graph Φ, with the gains listed on each edge.

In the example we contract the link e_4, giving the graph Φ/e_4. First, we switch the gains so that e_4's gain is the identity. In our example we will use the switching function defined by:

$$\zeta(v_1) = 1, \quad \zeta(v_1) = 1, \quad \zeta(v_3) = \frac{1}{15}.$$

The resulting gains are then

$$\begin{aligned}
\phi^\zeta(e_1) &= \zeta(v_1)^{-1}\phi(e_1)\zeta(v_2) = 1 \times 3 \times 1 = 3,\\
\phi^\zeta(e_2) &= \zeta(v_2)^{-1}\phi(e_3)\zeta(v_3) = 1 \times 5 \times \frac{1}{15} = \frac{1}{3},\\
\phi^\zeta(e_3) &= \zeta(v_3)^{-1}\phi(e_3)\zeta(v_1) = 15 \times 2 \times 1 = 30,\\
\phi^\zeta(e_4) &= \zeta(v_1)^{-1}\phi(e_4)\zeta(v_3) = 1 \times 15 \times \frac{1}{15} = 1.
\end{aligned}$$

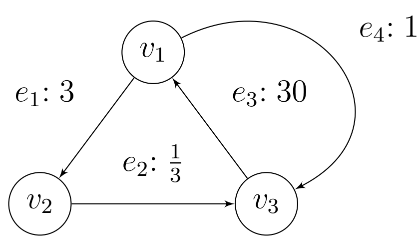

FIGURE 2. The gain graph Φ^ζ, with the switched gains.

For the second step, contract e_4, keeping all gains. The resulting gain graph, Φ/e_4, is shown in Figure 3. Notice that the digon in Φ/e_4 is

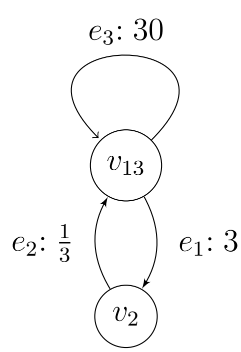

FIGURE 3. Φ/e_4; The vertex v_{13} is the coalescence of vertices v_1 and v_3.

balanced and that these edges came from the balanced circle $e_1e_2e_4^{-1}$ in Φ. In fact, when contracting edge e, every circle C through e becomes a new circle C/e that is balanced if, and only if, C was originally balanced. This is because the gains of circles do not change under switching and the gains of edges do not change when e is contracted.

Example 2. Next, we look at contraction of an unbalanced loop (or a half edge). Let $\Psi = \Phi/e_4$ from Example 1 and consider Ψ/e_3. The two endpoints of this loop coincide and the loop is unbalanced. We delete the edge and remove its incident vertex v_{13}. All edges incident with v_{13} lose that vertex; thus, e_1 and e_2 become half edges. The contraction Ψ/e_3 is shown in Figure 4.

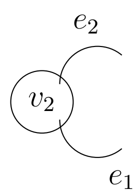

FIGURE 4. Ψ/e_3; Since the only remaining edges are half edges, there are no longer any gains.

There are no edges not incident with v_{13}, but if there were, they would retain their gains. If there were any half edges or loops incident to the deleted vertex, they would become loose edges; see Figure 5.

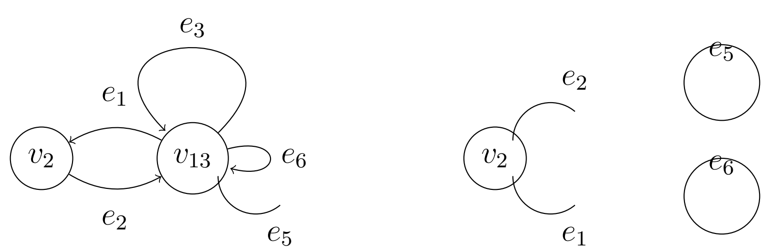

FIGURE 5. The left graph Ψ' is Ψ with two added edges. The gains of the added edges are irrelevant to the example. On the right is the contraction Ψ'/e_3, with two half edges and two loose edges. Those edges no longer have gains.

Chromatic polynomials.

Now we can return to chromatic polynomials.

An edge is said to be balanced if the edge, as an edge set, is balanced. That is, a link, a balanced loop, and a loose edge are balanced, while a half edge and an unbalanced loop are unbalanced.

Observe that, by their algebraic definitions (andwriting $\Omega = \langle \Phi \rangle$ for brevity), $\chi_\Phi = \chi_\Omega$ and $\chi^b_\Phi = \chi^b_\Omega$.

Theorem 1 (Deletion-Contraction for Chromatic Polynomials)**.** *For a gain graph Φ and an edge e,*

$$\chi_\Phi(\lambda) = \chi_{\Phi\setminus e}(\lambda) - \chi_{\Phi/e}(\lambda)$$

and

$$\chi^b_\Phi(\lambda) = \begin{cases} \chi^b_{\Phi\setminus e}(\lambda) - \chi^b_{\Phi/e}(\lambda) & \text{if } e \text{ is a balanced edge,} \\ \chi^b_{\Phi\setminus e}(\lambda) & \text{if } e \text{ is an unbalanced edge.} \end{cases}$$

The same is true for a biased graph Ω.

Proof. We will prove the theorem for biased graphs. The result for gain graphs follows directly because the polynomials are the same.

First, we prove it for the chromatic polynomial. The definition says $\chi_\Omega(\lambda) = \sum_{S\subseteq E}(-1)^{\#S}\lambda^{b(S)}$. We break the sum up into two parts, one for the edge sets that contain e and one for those that do not. Thus,

$$\chi_\Omega(\lambda) = \sum_{S\subseteq E\setminus e}(-1)^{\#S}\gamma^{b(S)} + \sum_{e\in S\subseteq E}(-1)^{\#S}\gamma^{b(S)}.$$

The first sum is $\chi_{\Omega\setminus e}(\lambda)$. For the second sum, we write each edge set containing e, S, as $T \cup e$ for some edge set $T \subseteq E \setminus e$. The sum then becomes $\sum_{T\subseteq E\setminus e}(-1)^{\#T+1}\lambda^{b_\Omega(T\cup e)}$. Note that the exponent is the number of balanced components in Ω, not Ω/e. We now have

$$\chi_\Omega(\lambda) = \chi_{\Omega\setminus e}(\lambda) - \sum_{T\subseteq E/e}(-1)^{\#T}\lambda^{b_\Omega(T\cup e)}$$

The sum over$T$ equals $\chi_{\Omega/e}(\lambda)$ by the following lemma, which completes the proof □

Lemma 1. $b_\Omega(T\cup e) = b_{\Omega/e}(T)$ *for any edge e and any set $T \subseteq E \setminus e$.*

Proof. The first case we look at is where e is a link. For that we apply the following lemma.

Lemma 2. *If Ω is balanced and $e \in E$, then Ω/e is also balanced.*

The proof of this lemma will be given at a later date.

If e is in a balanced component, B, then use Lemma 2.

If e is in an unbalanced component, U, we have three cases.

Case 1. e is a link. If e is in an unbalanced circle C, then C/e is an unbalanced circle in U/e. Hence U/e is unbalanced. Suppose the component has an unbalanced circle C such that C is a circle in U/e. Then C is unbalanced in U/e, so U/e is unbalanced. Suppose U has an unbalanced circle C of which e is a chord. Then $C \cup e$ contains two other circles, say C_1 and C_2. At least one of these must be unbalanced, say C_1. Then C_1/e is unbalanced in U/e. Therefore U/e is unbalanced in this case as well. Hence $b(\Omega/e) = b(\Omega)$.

The remaining cases will be proved in the next lecture.

□