Lecture 9: Chromatic Polynomial Counts Colorations

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Theorem 1 (Deletion-Contraction for Chromatic Polynomials)**.**

Proof for the balanced chromatic polynomial. The definition is

$$\begin{aligned}\chi^b_\Phi(\lambda) &= \sum_{\substack{S\subseteq E\\ S\text{balanced}}} (-1)^{\#S}\lambda^{b(S)}\\ &= \sum_{\substack{S\subseteq E\setminus e\\ S\text{ balanced}}} (-1)^{\#S}\lambda^{b(S)} - \sum_{\substack{T\subseteq E\setminus e\\ T\cup e\text{ balanced}}} (-1)^{\#T}\lambda^{b_\Omega(T\cup e)}.\end{aligned}$$

By Lemma **??**, the last summation equals

$$\sum_{\substack{T\subseteq E\setminus e\\ T\cup e\text{ balanced}}} (-1)^{\#T}\lambda^{b_{\Omega/e}(T)}$$

Since e is a balanced edge, by Lemma **??** the edge set T of Ω/e is balanced if and only if $T\cup e$ is balanced in Ω. Hence, the last summation

$$= \sum_{\substack{T\subseteq E(\Omega/e)\\ T\text{ balanced}}} (-1)^{\#T}\lambda^{b_{\Omega/e}(T)}$$

That concludes the proof. □

Switching of colorations. We define the switching γ^ζ of a coloration γ with respect to a switching function ζ to be

$$\gamma^\zeta(v) = \gamma(v)\zeta(v).$$

We see that

$$\begin{aligned}\gamma(v)\zeta(v)(\zeta(v)^{-1}\phi(e_{vw})\zeta(w)) = \gamma(v)\phi(e_{vw})\zeta(w) = \gamma(w)\zeta(w)\\ \iff \gamma(w) = \gamma(v)\phi(e_{vw}).\end{aligned} \tag{1}$$

Therefore a coloration is proper at a link if and only it is proper at the link after switching.

Define

$$K(\Phi) := \text{the set of proper } k\text{-colorations of } \Phi,$$

and similarly $K(\Phi\setminus e)$ = the set of proper k-colorations of $\Phi\setminus e$ and $K(\Phi/e)$ = the set of proper k-colorations of Φ/e. We are now obliged to mention something that we swept under the rug in defining contraction: the gains of Φ/e depend on the choice of the switching function ζ by which we switched e to have gain ϵ. Nevertheless, all possible contractions Φ/e are switching equivalent (that is an exercise for the reader), so it follows from Equation (1) that, although Φ/e depends on the choice of switching function, the resulting $K(\Phi/e)$'s are all bijective to each other by bijections that preserve the 0-colored set, and therefore their cardinalities are all the same.

Counting proper colorations. At last we can prove the main theorem about the chromatic polynomials.

Theorem 2. *Assume $\mathfrak{G}$ is finite and Φ is a $\mathfrak{G}$-gain graph of finite order. Let $m = |\mathfrak{G}|$. Then*

$$\chi_\Phi(km+1) = \text{the number of proper } k\text{-colorations of } \Phi, \text{ and}$$
$$\chi^b_\Phi(km) = \text{the number of zero-free proper } k\text{-colorations.}$$

We formulate the main parts of the proof as two lemmas. The theorem will follow by some special cases and induction on the number of edges.

For the first part of the theorem we define $\hat\chi_\Phi(km+1) :=$ the number of proper k-colorations of Φ. So $\hat\chi_\Phi$ is a function evaluated at positive integers of residue 1 (mod m).

Lemma 1. *If e is a link in Φ, then $\hat\chi_\Phi = \hat\chi_{\Phi\setminus e} - \hat\chi_{\Phi/e}$.*

Proof. Let e_{vw} be a link in Φ, also denoted more briefly by e. We simplify the proof by assuming Φ has been switched so e has gain ϵ. Then contraction of e does not require any switching.

A coloration is proper if and only if it is proper at each edge, i.e., $\gamma(b) \neq \gamma(a)\phi(e_{ab})$ for every edge e_{ab}. Since $\Phi \setminus e$ has all the vertices and edges of Φ except e,

$$K(\Phi) = \{\gamma \in K(\Phi \setminus e) \mid \gamma(w) \neq \gamma(v)\phi(e_{vw}) = \gamma(v)\}.$$

Consider a coloration that is proper except at e_{vw}. It gives a proper coloration of Φ/e because the color at v is the same as at w. In Φ/e, the color of the contraction vertex v_e is $\gamma(v)$; all other vertices retain the same color as in Φ.

Contrariwise, given a proper coloration $\hat\gamma$ of Φ/e, we construct a coloration γ of Φ by

$$\gamma(u) = \begin{cases} \hat\gamma(u) & \text{if } u \neq v, w, \\ \hat\gamma(v_e) & \text{if } u = v \text{ or } w, \end{cases}$$

where v_e is the contraction vertex. It is easy to see that these two constructions are inverse to each other, so they give a bijection between $\{\gamma \in K(\Phi \setminus e) \mid \gamma(w) = \gamma(v)\phi(e_{vw})\}$ and $K(\Phi/e)$.

Hence we have proved for the case when e is a link that there is a bijection

$$K(\Phi) \sqcup K(\Phi/e) \longleftrightarrow K(\Phi \setminus e).$$

It follows that $\hat{\chi}_\Phi = \hat{\chi}_{\Phi\setminus e} - \hat{\chi}_{\Phi/e}$. □

For the second part of the proof we define $\hat{\chi}^b_\Phi(km) :=$ the number of zero free proper k-colorations of Φ. So $\hat{\chi}^b_\Phi$ is a function evaluated at nonnegative integer multiples of m.

Lemma 2. *If e is a link in Φ, then $\hat{\chi}^b_\Phi = \hat{\chi}^b_{\Phi\setminus e} - \hat{\chi}^b_{\Phi/e}$.*

We will prove this lemma in the next lecture.