Relative Vertices are Translates

We assume $\mathscr{A}$ is an affine hyperplane arrangement in $\mathbb{A}^n(K)$, K a field. Its projectivization is $\mathscr{A}_\mathbb{P} := \{h_\mathbb{P} : h \in \mathscr{A}\} \cup \{h_\infty\}$ in $\mathbb{P}^n(K)$, where for an affine flat s, $s_\mathbb{P}$ denotes its extension to a projective flat by adjoining the necessary ideal points, and h_∞ is the ideal hyperplane.

The relative vertices are the maximal elements of $\mathscr{L}(\mathscr{A})$; that is, the affine intersection flats that are minimal in the containment ordering of affine flats.

Theorem 1. *All relative vertices of an affine arrangement $\mathcal{A}$ are translates of one another. In fact, every relative vertex v of $\mathcal{A}$ satisfies $v_\mathbb{P} \cap h_\infty = \hat{1} := \bigcap \mathcal{A}_\mathbb{P}$.*

Proof. [1] We implicitly use the modular law of dimension in projective space, several times.

Let r be the rank of $\mathcal{L}(\mathcal{A}_\mathbb{P})$, and assume by way of contradiction that v is a relative vertex of rank at most $r-2$ (i.e., $v_\mathbb{P} \ll \hat{1}$); that is, $v_\mathbb{P}$ is covered by $v_\infty := v_\mathbb{P} \cap h_\infty$, which itself is covered by some ideal flat t_∞.

We have two ways to complete the result.

Geometrical Proof.[2] We claim $t_\infty = t_\mathbb{P} \cap h_\infty$ for some $t \in \mathcal{L}(\mathcal{A})$, which means that $t > v$ in $\mathcal{L}(\mathcal{A})$, contradicting the choice of v as a relative vertex. To see this, let $h \in \mathcal{L}(A)$ such that $t_\infty = v_\mathbb{P} \cap h_\mathbb{P}$. Define $t := h \cap v$. Then $t \in \mathcal{L}(\mathcal{A})$ and $t > v$, so long as $t \neq \varnothing$. If $t = \varnothing$, then $h_\mathbb{P} \cap v_\mathbb{P} \subseteq h_\infty$, so $h_\mathbb{P} \cap v_\mathbb{P} = h_\mathbb{P} \cap v_\mathbb{P} \cap h_\infty = h_\mathbb{P} \cap v_\infty = t_\infty$, a contradiction. Thus $h_\mathbb{P} \cap v_\mathbb{P} \not\subseteq h_\infty$, so $t_\mathbb{P} = t_\infty$ and $t \neq \varnothing$. Therefore $v_\mathbb{P} \lessdot \hat{1} = v_\mathbb{P} \cap h_\infty$. This means that all relative vertices v have the same ideal part, namely $\hat{1}$, and we are done.

Geometric Lattice Proof. By atomicity of $\mathcal{L}(\mathcal{A}_\mathbb{P})$, there is a hyperplane $h_\mathbb{P} \in \mathcal{A}_\mathbb{P}$ such that $t_\infty = v_\infty \wedge h_\mathbb{P}$. Then $h_\mathbb{P} \not\leq v_\mathbb{P}$ (since $t_\infty > v_\infty$) so $h_\mathbb{P} \wedge v_\mathbb{P} > v_\mathbb{P}$ and by semimodularity $v_\mathbb{P} \wedge h_\mathbb{P} \lessdot t_\infty$. Since $v_\mathbb{P} \wedge h_\mathbb{P} = v_\infty$ leads to the contradiction that $t_\infty = v_\infty$, $v_\mathbb{P} \wedge h_\mathbb{P}$ is a projective flat of $\mathcal{A}_\mathbb{P}$ but not an idea flat. It therefore is the projective extension of an affine flat t of $\mathcal{A}$, which is $> v$. This contradiction shows that $v_\mathbb{P} \lessdot \hat{1}$ and $v_\infty = \hat{1}$, which implies that all relative vertices have the same ideal part, which in turn implies they are translates of each other. □

It is recommended to draw a Hasse diagram to illustrate the proofs.

[1] Finally finished by T.Z., 29 Nov. 2019

[2] Based on notes by Andrew Lamoureux from T.Z. lecture.