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ARRANGEMENTS OF HYPERPLANES; MATROIDS AND GRAPHS

by

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An arrangement of hyperplanes is a finite set of hyperplanes in Euclidean d-space. (Or in projective space; which can be treated as the Euclidean "central" case.) To see some arrangements of lines in $\mathbb{E}^2$ and of planes in $\mathbb{E}^3$, turn to the end of this article.

What one is interested in about an arrangement $\mathcal{H}$ is the decomposition of the space due to the hyperplanes. When they are deleted the remainder of the space falls into components, called the regions of $\mathcal{H}$. The closure of a region is a d-dimensional polyhedron (not necessarily bounded); the k-faces of $\mathcal{H}$ for each dimension $0 \le k \le d$ are the k-dimensional faces of these polyhedra. The flats of $\mathcal{H}$ are the non-void subspaces obtained by taking intersections of hyperplanes. (For projective arrangements the empty flat is not excluded.) Here are some typical questions: How many regions are there? How many k-faces? How many are bounded? How many are shaped like simplices? What is the topology of the bounded faces?

All these questions are purely combinatorial: they involve only the incidence relations among the faces and not, for example, angles or volumes. But although apparently similar, they differ sharply in how much one must know about the arrangement in order to answer them. To count simplicial regions or to describe exactly the bounded topology it is apparently necessary to know the full combinatorial structure of $\mathcal{H}$. But to count regions or faces or bounded regions and faces, curiously enough, much less is required: what one needs to know is merely the partially ordered set of flats of $\mathcal{H}$. (See Figure 4.) It is this fact and its consequences that I will discuss.

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Face-count formulas.

The set of flats, ordered by reverse inclusion (so its least element 0 is the whole space $= \cap\, \emptyset$), we denote $\mathcal{L}(\mathcal{H})$ and call the semilattice of flats (in [10], "cut-intersection semilattice") of $\mathcal{H}$. Because of its reverse ordering $\mathcal{L}(\mathcal{H})$ is a lower ideal of a geometric lattice (and is a geometric lattice if $\mathcal{H}$ is projective or central (Figures 2, 5)); thus matroids appear in the enumerations.

The formulas employ Rota's Möbius function $\mu(s,t)$, defined on ordered pairs of flats of $\mathcal{H}$. Here is a definition which is equivalent to the usual recursive one but suggests the inclusion-exclusion character of μ. First for $s = 0$ (needed to count regions):

$$\mu(0,t) = \sum_{\mathcal{F} \subseteq \mathcal{H}:\ \cap \mathcal{F} = t} (-1)^{\#\mathcal{F}} .$$

More generally, let $\mathcal{S} = \{h \in \mathcal{H} \mid h \supseteq s\}$; then

$$\mu(s,t) = \sum_{\substack{\mathcal{S} \subseteq \mathcal{F} \subseteq \mathcal{H} \\ \cap \mathcal{F} = t}} (-1)^{\#\mathcal{F} - \#\mathcal{S}} ,$$

The numbers of regions and faces result from evaluating the characteristic polynomial of $\mathcal{H}$,

$$p_{\mathcal{H}}(y) = \sum_{t \in \mathcal{L}(\mathcal{H})} \mu(0,t)\, y^{r - \operatorname{codim} t} ,$$

where r is the rank of $\mathcal{H}$: the maximum codimension of any flat; and the Whitney polynomial ("Möbius polynomial" in [10]),

$$w_{\mathcal{H}}(x,y) = \sum_{s,t \in \mathcal{L}(\mathcal{H})} x^{\operatorname{codim} s} \mu(s,t)\, y^{r - \operatorname{codim} t} .$$

We write $f_k(\mathcal{H})$ = the number of k-faces of $\mathcal{H}$.

Theorem A ([10]). Let $\mathcal{E}$ be a Euclidean arrangement with rank r. The number of its regions is

$$c(\mathcal{E}) = (-1)^r p_{\mathcal{E}}(-1) = \sum_{t \in \mathcal{L}(\mathcal{E})} |\mu(0,t)| .$$

Let $f_{\mathcal{E}}(x) = \Sigma_{k=0}^{d} f_k(\mathcal{E})\, x^{d-k}$. Then

$$f_{\mathcal{E}}(x) = (-1)^r w_{\mathcal{E}}(-x,-1) .$$

Theorem B ([10]). Let $\mathcal{P}$ be a projective arrangement with rank r. (Assume $\mathcal{P} \neq \emptyset$.) The number of its regions is

$$c(\mathcal{P}) = \frac{1}{2} (-1)^r p_{\mathcal{P}}(-1) = \frac{1}{2} \sum_{t \in \mathcal{L}(\mathcal{P})} |\mu(0,t)| .$$

The polynomial $f_{\mathcal{P}}(x) = \Sigma_{k=-1}^{d} f_k(\mathcal{P})\, x^{d-k}$ satisfies

$$f_{\mathcal{P}}(x) = \frac{1}{2} [x^r + (-1)^r w_{\mathcal{P}}(-x,-1)] .$$

Two dual versions of Theorem B, in [10], Sec. 6, describe the number of inequivalent ways to put a hyperplane between the points of a finite subset of Euclidean or projective space, and count the faces of zonotopes.

The next theorem requires a little explanation. When the rank of $\mathcal{E}$ is less than d, the smallest flats of $\mathcal{E}$ (called its relative vertices) are not points; their dimension is $d - \operatorname{rk} \mathcal{E}$. Thus there can be no bounded faces. But suppose we cross-section the faces by a subspace t, whose dimension is $\operatorname{rk} \mathcal{E}$, perpendicular to the relative vertices. Then the cross-section of a relative vertex is a point. Call a face of $\mathcal{E}$ relatively bounded if its cross-section is bounded.

Theorem C ([10]). The number of relatively bounded regions of a Euclidean arrangement $\mathcal{E}$ with rank r is

$$c^{bd}(\mathcal{E}) = (-1)^r p_{\mathcal{E}}(1) = \Big| \sum_{t \in \mathcal{L}(\mathcal{E})} \mu(0,t) \Big| .$$

The polynomial $f^{bd}_{\mathcal{E}}(x) = \Sigma_{k=0}^{d} f^{bd}_k(\mathcal{E})\, x^{d-k}$ satisfies

$$f^{bd}_{\mathcal{E}}(x) = (-1)^r w_{\mathcal{E}}(-x,1) .$$

Corollary C1 (from [10], Cor. 2.2. See Figure 5(c)). Let $\mathcal{H}$ be a central Euclidean arrangement and g a hyperplane parallel to (and not containing) $z = \bigcap \mathcal{H}$ but otherwise in general position with respect to $\mathcal{H}$. Then $|\mu(0,z)|$ equals the number of relatively bounded regions of $\mathcal{H} \cup \{g\}$, or also of $\mathcal{H}_g$.

The importance of this corollary was realized by Curtis Greene. Suppose v is a vertex of a Euclidean arrangement $\mathcal{S}$, $\mathcal{H} = \{h \in \mathcal{S} \mid h \supseteq v\}$, and g lies in general position with respect to $\mathcal{S}$. Then if g sweeps from one side of v to the other, some bounded regions of $\mathcal{S}_g$ disappear at v and reappear on the other side--but now they are different regions of $\mathcal{S}$. By Corollary C1, g has met $|\mu(0,v)|$ new regions of $\mathcal{S}$. This fact provides the key to generalizing to dimensions higher than 3 the sweep hyperplane counting technique developed by Wetzel and his associates (in [1] and [2] among other papers) from an idea of Brousseau.

Theorems A-C have as corollaries the well-known Euler relations, for evaluating at $x = -1$ gives $(-1)^{d-r}$ times the Euler number (that is, $f_0 - f_1 + \ldots \pm f_d$) of the appropriate d-space (Euclidean, projective, or bounded). This number in the Euclidean and projective cases is known independently from topology and can form the starting point for a proof of the face-count formulas by Möbius inversion. That was essentially Buck's way of handling simple arrangements. But proving the Euler number of the bounded space equals $(-1)^{d-r}$ is not trivial; Buck failed to show it and I know of no proof independent of Theorem C.

A second proof, entirely independent of the Euler relations, is based on the theory of Tutte-Grothendieck invariants of matroids developed by Brylawski. If an arrangement $\mathcal{H}$ partitions into subsets $\mathcal{A}$ and $\mathcal{B}$ such that $\operatorname{rk} \mathcal{H} = \operatorname{rk} \mathcal{A} + \operatorname{rk} \mathcal{B}$, then $\mathcal{H}$ is called the direct sum of $\mathcal{A}$ and $\mathcal{B}$, written $\mathcal{A} \oplus \mathcal{B}$. (This means there are subspaces a and b meeting in a single point such that every $\mathcal{A}$ hyperplane is perpendicular to a and every $\mathcal{B}$ hyperplane to b. See Figure 2.) If t is any subspace of the whole space, let

$$\mathcal{H}_t = \text{the arrangement induced by } \mathcal{H} \text{ on } t = \{g \cap t \mid g \in \mathcal{H} \text{ and } \dim(g \cap t) = \dim t - 1\}.$$

A function of arrangements having the properties

$$f(\mathcal{A}) = f(\mathcal{B}) \text{ if } \mathcal{L}(\mathcal{A}) \cong \mathcal{L}(\mathcal{B}),$$

$$f(\mathcal{A} \oplus \mathcal{B}) = f(\mathcal{A})\, f(\mathcal{B}),$$

$$f(\mathcal{H}) = f(\mathcal{H} \setminus h) + f(\mathcal{H}_h) \text{ if } h \in \mathcal{H} \text{ is not a summand of } \mathcal{H},$$

is called a Tutte-Grothendieck invariant. Two T-G invariants which agree in value on the empty arrangement and on the arrangement of a single hyperplane $\{h\}$ must agree on all arrangements. It can be shown that $(-1)^r p_{\mathcal{H}}(y)$ and $c(\mathcal{H})$ are T-G invariants. Since $p_{\{h\}}(y) = y - 1$ and $\operatorname{rk}\{h\} = 1$,

$$c(\{h\}) = 2 = (-1)^{\operatorname{rk}\{h\}} p_{\{h\}}(-1).$$

Theorems A and B follow. Theorem C is then provable from Theorem B by using the projectivization of $\mathcal{S}$: the projective arrangement

$$\mathcal{S}_{\mathbb{P}} = \{h_\infty\} \cup \{h_{\mathbb{P}} \mid h \in \mathcal{S}\},$$

where h_∞ = the hyperplane at infinity in $\mathbb{P}^d$ and $h_{\mathbb{P}}$ = the completion of h in projective space. The details of all these proofs appear in [10].

Theorems A-C and their proofs (except part of that of Lemma 4C2 in [10]) apply more generally: they hold for oriented matroids; thus, by an observation of Jim Lawrence, for arrangements of "topological hyperplanes", which are allowed to wiggle, so as for instance to violate Desargues's theorem, but must still cross wherever they intersect and in other ways behave topologically like flat hyperplanes. Incidentally I was led to study arrangements of hyperplanes in 1971 from work on axioms for oriented matroids, so the relationship is not unexpected.

Some special cases.

It is easy to see that $\mu(0,0) = 1$, $\mu(0,h) = -1$ for any hyperplane h, and $|\mu(s,t)| = 1$ whenever $s \leq t$ in a simple arrangement. And if in an arrangement of lines v is a vertex lying on n_v lines and ℓ is a line through v, then $\mu(0,v) = n_v - 1$ and $\mu(\ell,v) = -1$. Thus the classical planar formulas, known at least since 1966 (see [3]), and the formulas for simple d-dimensional arrangements (see Figure 3), known since Buck's article and in part earlier, are special cases of the general results. In the same way so are the "additive" formulas for $\mathbb{E}^3$ deduced in [2] by the sweep plane method.

Annother special case would, if it could be solved explicitly, yield the number of threshold functions. A threshold function of n inputs is a switching function, $f: \{\pm 1\}^n \to \{\pm 1\}$, which can be expressed in the linear form

$$f(x_1,x_2,\ldots,x_n) = \mathrm{sgn}(a_0 + a_1x_1 + \ldots + a_nx_n) .$$

The distinct threshold functions correspond to the regions of a certain central arrangement $\mathcal{J}_n$ in $\mathbb{R}^{n+1}$, which means they can be counted--in principle. Furthermore (I have been told), good estimates of the numbers of k-faces of $\mathcal{J}_n$ would tell something about the computational complexity of the knapsack problem. Unfortunately the lattice $\mathcal{L}(\mathcal{J}_n)$ is so complicated it has not yet been possible to analyze it.

Bounded regions and a decomposition of arrangements.

The projectivization of a Euclidean arrangement leads us to the first enumerative interpretation of the β invariant introduced by Crapo [4] for matroids. For a projective arrangement $\mathcal{P}$,

$$\beta(\mathcal{P}) = (-1)^r \sum_{t \in \mathcal{L}(\mathcal{P})} \mu(0,t)\,[d - \dim t] .$$

It follows from [4] that β is nonnegative and that it is 0 on direct sums and on no other arrangements.

Theorem D ([10]). The number of relatively bounded regions of a Euclidean arrangement $\mathcal{E}$ is

$$c^{bd}(\mathcal{E}) = \beta(\mathcal{E}_{\mathbb{P}}) .$$

Corollary D1 ([10], Cor. 7.1). $\mathcal{E}$ has a relatively bounded region iff $\mathcal{E}_{\mathbb{P}}$ is not a direct sum.

$\mathcal{E}$ is derived from the projective arrangement $\mathcal{P} = \mathcal{E}_{\mathbb{P}}$ by choosing a hyperplane $h \in \mathcal{P}$ to be at infinity. A different choice of h will lead to a new arrangement $\mathcal{P} \setminus \{h\}$ in the Euclidean space $\mathbb{P}^d \setminus h$, with the same number of hyperplanes and the same regions as $\mathcal{E}$. Obviously which regions are bounded will change with h. Nevertheless,

Corollary D2 ([10], Cor. 7.3. See Figure 6). The number of bounded regions of a Euclidean arrangement derived from $\mathcal{P}$ is independent of the choice of $h \in \mathcal{P}$ to be at infinity.

We could look at the projectivization in another way by pulling back from $\mathbb{P}^d$ to $\mathbb{R}^{d+1}$. Then $\mathcal{E}_{\mathbb{P}}$ becomes a central arrangement $\mathcal{H}$ in which the infinite hyperplane is $h_0 = \{x \mid x_0 = 0\}$, the original affine space is $a = \{x \mid x_0 = 1\}$, and $\mathcal{E}$ is the induced arrangement $\mathcal{H}_a$. The relatively bounded regions of $\mathcal{E}$ are facets of those of $\mathcal{H} \cup \{a\} \setminus \{h_0\}$. From Theorem D:

Corollary D3. (See Figure 5(d).) Suppose $\mathcal{H}$ is a central Euclidean arrangement and $h \in \mathcal{H}$. Let h^* be h perturbed by translation from its initial position and let $\mathcal{H}^*$ be the perturbed arrangement with or without h also. The number of relatively bounded regions of $\mathcal{H}^*$, or also of the induced arrangement $\mathcal{H}_{h^*}$, is equal to $\beta(\mathcal{H})$ --regardless of the choice of h.

We can also extract from Theorem D a nice geometric criterion for the existence of a bounded region (Corollary E1). Although the criterion is no more than obvious, its proof seems to require the full strength of the following theorem, whose proof depends on Theorem D and is not trivial. I state it for the rank d case.

Theorem E ([10]). Assume the Euclidean arrangement $\mathcal{E}$ has rank d. Let z be the affine hull of the vertices, $\mathcal{Z} = \{h \in \mathcal{E} \mid h \supseteq z\}$, and $\mathcal{P} = \mathcal{E} \setminus \mathcal{Z}$. Then

$$\mathcal{E} = \mathcal{Z} \oplus \mathcal{P} ,$$

$\cap \mathcal{Z} = z$,

$\mathcal{P}$ is isomorphic (as an incidence structure of faces and hyperplanes) to $\mathcal{P}_z$, its cross-section by z , and

$\mathcal{P}_z$ has one or more bounded regions.

Thus $\mathcal{P}$ is the largest subarrangement which has a (relatively) bounded region. That yields the promised criterion:

Corollary E1 ([10], Cor. 8.1). A Euclidean arrangement has a bounded region if and only if its vertices span $\mathbb{E}^d$.

For more about the bounded part of a Euclidean arrangement see [10], Sec. 9.

Arrangements and graphs.

Many of the results I have described can be applied to graphs by way of the graphic arrangement. To Γ , a graph without loops on the nodes $p_1, p_2, \ldots, p_d$, corresponds the arrangement

$$\mathcal{H}[\Gamma] = \{h_{ij} \mid \Gamma \text{ has an edge } e_{ij}\}$$

in $\mathbb{R}^d$, where h_{ij} is the hyperplane defined by $x_i = x_j$. Two facts of matroid theory connect Γ and $\mathcal{H}[\Gamma]$: the chromatic polynomial $\chi_\Gamma(y)$ equals $y^c\, p_{\mathcal{H}[\Gamma]}(y)$, where c is the number of components of Γ ; and $\mathcal{L}(\mathcal{H}[\Gamma])$ is naturally isomorphic to the geometric lattice $\mathcal{L}(\Gamma)$ whose members are the circle-closed edge sets of Γ . The special value of $\mathcal{H}[\Gamma]$ comes from an observation of Crutis Greene: the regions of $\mathcal{H}[\Gamma]$ correspond one-to-one to the acyclic orientations of Γ . To construct the orientation $\alpha(R)$ from a region R , take a point $x \in R$ and orient e_{ij} from p_i to p_j if $x_i < x_j$. The graphic corollary of Theorem A is immediate. Here μ_Γ is the Möbius function of $\mathcal{L}(\Gamma)$. (See the illustrations in Figure 7.)

Theorem G1 (Stanley [9]). The number of acyclic orientations of Γ is

$$|\chi_\Gamma(-1)| = \sum_{T \in \mathcal{L}(\Gamma)} |\mu_\Gamma(0,T)| .$$

This result can be considerably refined. Greene has worked out an interpretation of each summand, too complicated to describe here. But one can easily interpret $\beta(\Gamma)$ by means of Corollary D3, choosing an edge e_{ij} and letting $\mathcal{H} = \mathcal{H}[\Gamma]$, $h = h_{ij}$, and $h^* = \{x \mid x_j = x_i + 1\}$. In any region which meets h^* , e_{ij} is oriented $p_i \to p_j$. A region R^* of $\mathcal{H}^*$ is relatively bounded if no x_k can go to $\pm\infty$ in R^* unless $\Sigma_i x_i \to \pm\infty$; this is the case precisely when no p_k except p_i and p_j is a source or sink in $\alpha(R^*)$.

Theorem G2 (Greene-Zaslavsky; see [7]). Let e_{ij} be any edge of Γ . The number of acyclic orientations of Γ in which p_i is the only source and p_j is the only sink is equal to

$$\beta(\Gamma) = |\chi'_\Gamma(1)| = \Big|\sum_{T \in \mathcal{L}(\Gamma)} \mu_\Gamma(0,T)\, c(T)\Big|$$

(where $c(T)$ = the number of components of T), which is positive if Γ is 2-connected and 0 otherwise.

In addition, $|\mu_\Gamma(0,1)|$ can be interpreted by means of Corollary C1. Choose any node p_i , let $z = \{x \mid x_1 + \ldots + x_d = 0\}$, and set $\mathcal{H} = \mathcal{H}[\Gamma]_z$ and $g = \{x \mid x_i = 1\}$.

Theorem G3 (Greene). Let p_i be any node of Γ . The number of acyclic orientations of Γ in which p_i is the only source is equal to $|\chi'_\Gamma(0)|$, which is the positive number $|\mu_\Gamma(0,1)|$ if Γ is connected, 0 otherwise.

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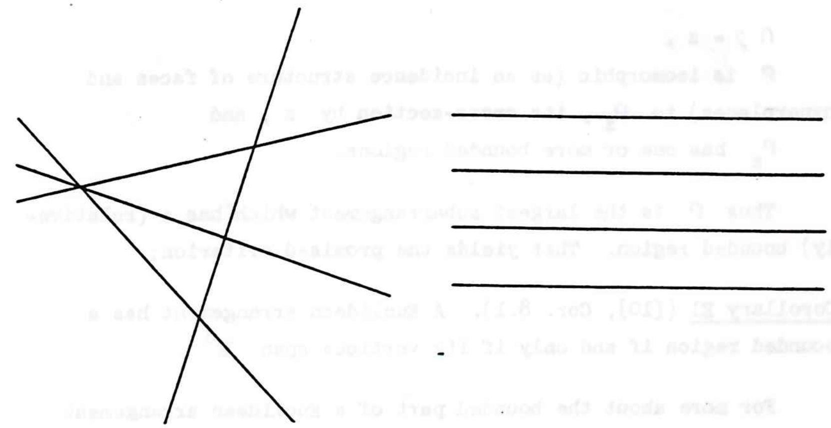


10 regions, 2 bounded.
13 edges, 5 bounded.
4 vertices, all bounded.

$p(y) = y^2 - 4y + 5.$

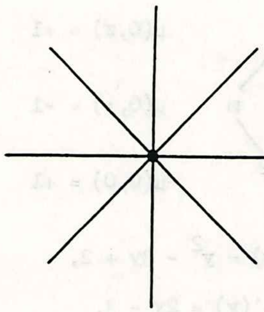
(a) An arrangement of 4 crossing lines. It has rank 2 - 0 = 2, since its smallest flats are vertices.

5 regions, 3 relatively bounded.
4 edges (= relative vertices).

$p(y) = y - 4.$

(b) An arrangement of 4 parallel lines. It has rank 2 - 1 = 1, since its smallest flats are 1-dimensional.

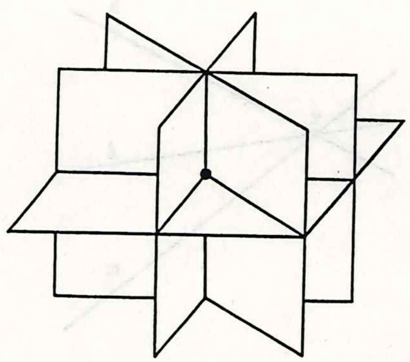
Figure 1. Two arrangements of lines in $\mathbb{E}^2$, their face numbers, and their characteristic polynomials.



8 regions,
8 edges,
1 vertex,
the only bounded face.

$$p(y) = y^2 - 4y + 3.$$

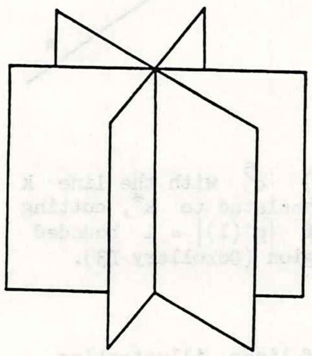
(a) A central arrangement of 4 lines. Its rank is 2 - 0 = 2.



12 regions,
18 facets,
8 edges,
1 vertex, the only bounded face.

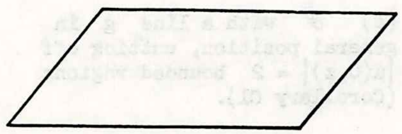
$$p(y) = y^3 - 4y^2 + 5y - 2$$
$$= p_{vert}(y)\, p_{horiz}(y).$$

(b) A central arrangement of 4 planes having rank 3 - 0 = 3. It is the direct sum of its vertical planes and its horizontal plane.



$$p_{vert}(y) = y^2 - 3y + 2.$$

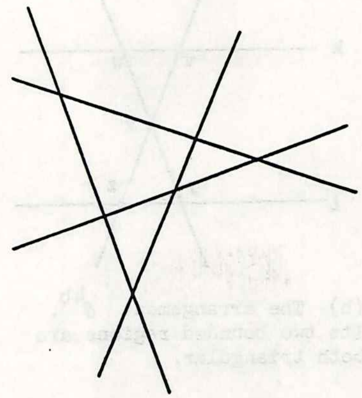
(c) The vertical planes of 2(b), having rank 3 - 1 = 2.



$$p_{horiz}(y) = y - 1.$$

(d) The horizontal plane of 2(b), having rank 3 - 2 = 1.

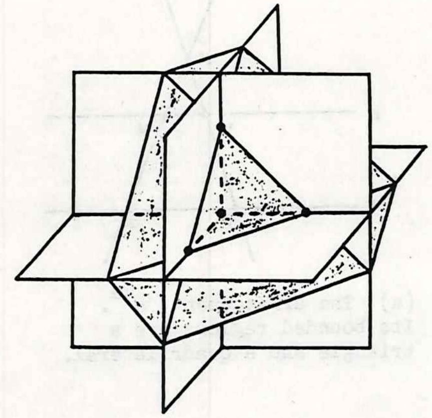
Figure 2. Four Euclidean arrangements which are central: the intersection of all the hyperplanes is nonempty. The decomposition of (b) into (c) and (d) illustrates direct sums.



11 regions, 3 bounded.
16 edges, 8 bounded.
6 vertices, all bounded.

$$p(y) = y^2 - 4y + 6.$$

(a) A simple arrangement of 4 lines.

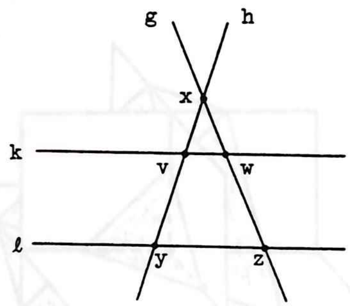


15 regions, 1 bounded.
28 facets, 4 bounded.
18 edges, 6 bounded.
4 vertices, all bounded.

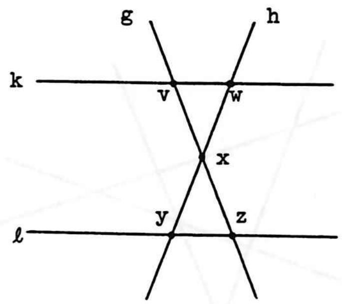
$$p(y) = y^3 - 4y^2 + 6y - 4.$$

(b) A simple arrangement of 4 planes.

Figure 3. Two Euclidean arrangements which are simple: the intersection of any k hyperplanes (for $k \le d + 1$) has dimension $d - k$.



(a) The arrangement $\mathcal{S}^{4a}$. Its bounded regions are a triangle and a quadrilateral.

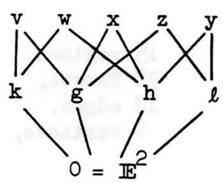


(b) The arrangement $\mathcal{S}^{4b}$. Its two bounded regions are both triangular.

rank 2   All $\mu(0,\cdot) = +1$

rank 1   All $\mu(0,\cdot) = -1$

rank 0 $\mu(0,0) = +1$



(c) The semilattice $\mathcal{L}$ of flats of $\mathcal{S}^{4a}$ and of $\mathcal{S}^{4b}$, and the Möbius function $\mu(0,t)$ for $t \in \mathcal{L}$. (It is not always true that $\mu(0,t)$ depends only on the rank of t.)

$p_{\mathcal{S}}(y) = y^2 - 4y + 5,\qquad w_{\mathcal{S}}(x,y) = y^2 - 4y + 5 + (4y - 10)x + 5x^2.$

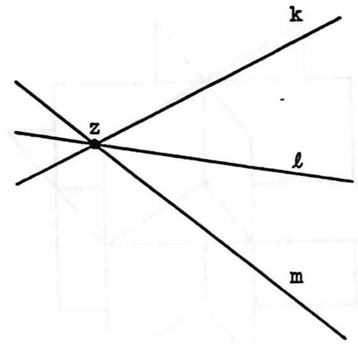
$|p_{\mathcal{S}}(-1)| = 10,\qquad f_{\mathcal{S}}(x) = (-1)^2 w_{\mathcal{S}}(-x,-1) = 5x^2 + 14x + 10,$

$|p_{\mathcal{S}}(1)| = 2,\qquad f^{bd}_{\mathcal{S}}(x) = (-1)^2 w_{\mathcal{S}}(-x, 1) = 5x^2 + 6x + 2.$

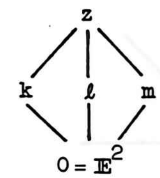
↑ vertices ↑ edges ↑ regions

(d) The characteristic and Whitney polynomials of both arrangements and their evaluations at -1 (to count faces) and $+1$ (to count bounded faces).

Figure 4. Two arrangements of 4 lines. Although not isomorphic (since only one has a quadrilateral region), they have the same semilattice of flats and consequently the same numbers of faces and bounded faces in each dimension.



(a) The arrangement $\mathcal{S}^5$.



$\mu(0,z) = +1$

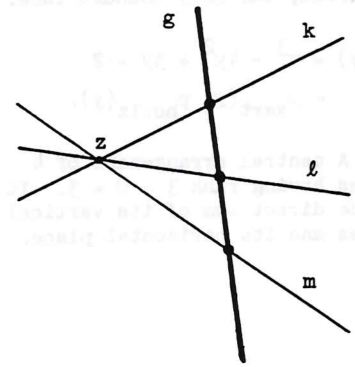
$\mu(0,\cdot) = -1$

$\mu(0,0) = +1$

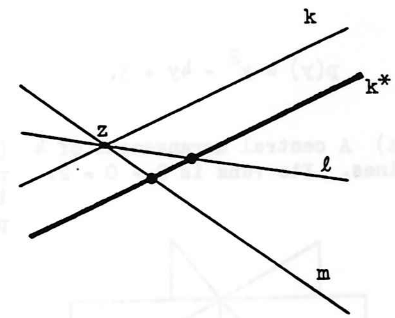
$p(y) = y^2 - 3y + 2,$

$p'(y) = 2y - 3.$

(b) The lattice $\mathcal{L}(\mathcal{S}^5)$ and its Möbius function.

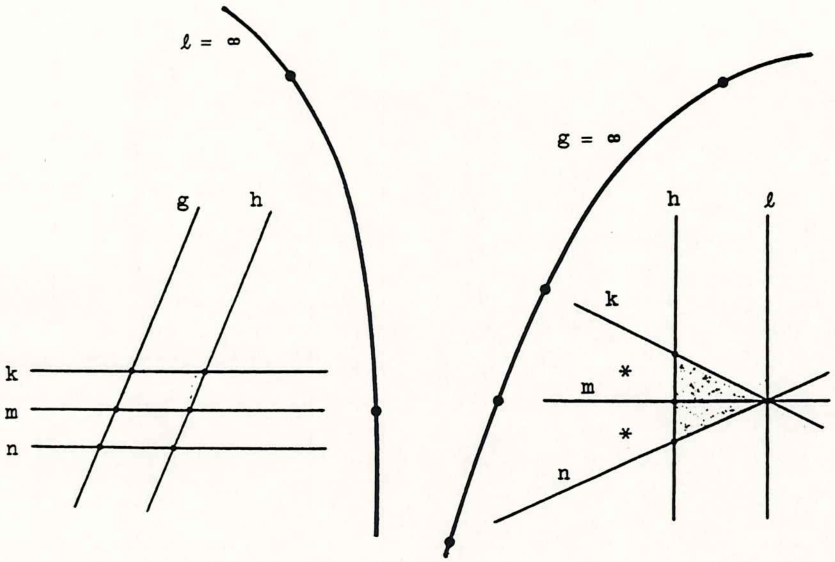


(c) $\mathcal{S}^5$ with a line g in general position, cutting off $|\mu(0,z)| = 2$ bounded regions (Corollary C1).



(d) $\mathcal{S}^5$ with the line k translated to k^*, cutting off $|p'(1)| = 1$ bounded region (Corollary D3).

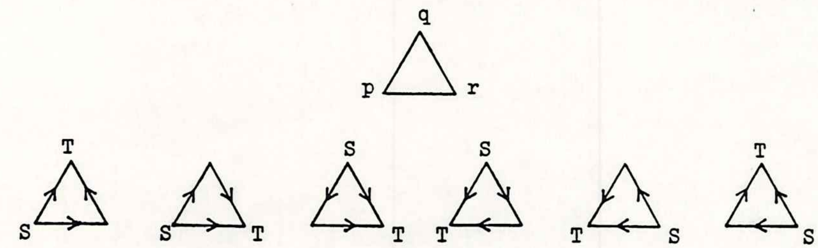
Figure 5. A central arrangement of lines, illustrating Corollary C1, in (c), and Corollary D3, in (d).



(a) The original arrangement $\mathcal{A}^{5a}$ of 5 lines in $\mathbb{E}^2$. $\mathcal{A}^{5a}_{\mathbb{P}}$, its projectivization, is the arrangement in $\mathbb{P}^2$ which includes the infinite line ℓ.

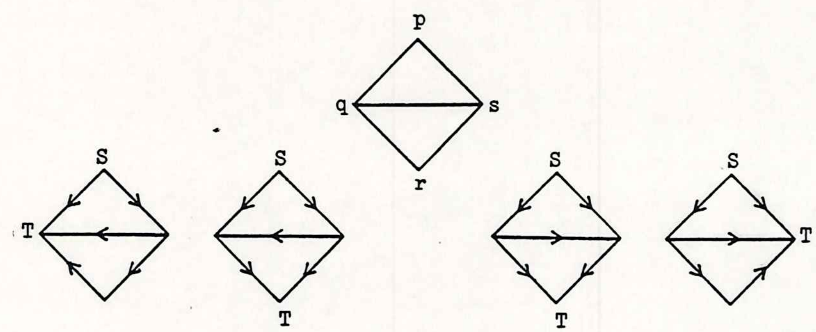
(b) The transformed arrangement $\mathcal{A}^{5b}$, with g thrown to infinity. The former bounded regions are starred. The projectivization $\mathcal{A}^{5b}_{\mathbb{P}}$ equals $\mathcal{A}^{5a}_{\mathbb{P}}$.

Figure 6. Arrangements of lines illustrating change of infinity (Corollary D2). Bounded regions are shaded.



$$\chi(y) = y(y - 1)(y - 2) = y^3 - 3y^2 + 2y$$

(a) The graph K_3 in all its $|\chi(-1)| = 6$ acyclic orientations. In $|\chi'(0)| = 2$ of them p is the sole source. In $|\chi'(1)| = 1$ orientation, q is also the only sink.



$$\chi(y) = y^4 - 5y^3 + 8y^2 - 4y$$

(b) A graph, showing all $|\chi'(0)| = 4$ acyclic orientations in which p is the only source. For each edge pp' there is $1 = |\chi'(1)|$ such orientation in which p' is the only sink.

Figure 7. Illustrations of Theorems G1-G3: acyclically oriented graphs with sources (S) and sinks (T) marked.