DUALITY
MATH 580, SPRING 2008

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I. Abstract Duality

(1) Duality in terms of closures. (Assume all sets are finite.)
   (a) Given a matroid M, for all $S \subseteq E$ and $x \in E \setminus S$, define $T = E \setminus S \setminus x$. Then either $x \in \operatorname{cl}(S)$ or $x \in \operatorname{cl}^*(T)$, but not both.
   (b) Given two closure operators on a set E, such that for all $S \subseteq E$ and $x \in E \setminus S$, either $x \in \operatorname{cl}(S)$ or $x \in \operatorname{cl}^*(E \setminus S \setminus x)$, but not both. Then cl and cl^* are the closure operators of a dual pair of matroids.

(2) I propose this problem as a group research project: Figure out a truly self-dual axiom system based on circuits and cocircuits and the property

$$|C \cap C^*| \neq 1 \quad \forall C \in \mathcal{C},\ C^* \in \mathcal{C}^*.$$

II. Duality of Vector Representations

Theorems of concern here are:

- *Theorem 2.2.A.* Given matrices A, A' over any field, with n columns. If they have the same row space, $\mathcal{R}(A') = \mathcal{R}(A)$, then they have the same column matroid: $M[A] = M[A']$.
- *Theorem 2.2.D.* Given matrices A, A^* over any field, with n columns. If $\mathcal{R}(A^*) = \mathcal{R}(A)^\perp$, then $M[A^*] = M^*[A]$.
- Theorem 2.2.8.
- *Theorem 2.2.G (Whitney's Orthogonality Theorem).* In the Euclidean vector space $\mathbb{R}^n$ (with dot product), let $b_1, \ldots, b_n$ be an orthonormal basis and let W be a subspace. Let y_i be the orthogonal projection of b_i onto W and let z_i be its orthogonal projection onto $W^\perp$. Let M be the vector matroid of $y_1, \ldots, y_n$. Then the vector matroid of $z_1, \ldots, z_n$ is M^*.

(1) A theorem of linear algebra states: Suppose you have a subspace W of $\mathbb{R}^n$ that is generated by a basis $\alpha_1, \ldots, \alpha_r$ and you form the matrix A whose rows are the vectors α_i, $i = 1, \ldots, r$. Then the orthogonal projection onto W of any vector $x \in \mathbb{R}^n$ is given by the formula $\operatorname{proj}_W x = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}Ax$. Find out how to prove this formula, either by looking it up or by working it out yourself.
(2) Make the assumptions of Theorem 2.2.G with the exception that $\{b_1, b_2, \ldots, b_n\}$ is the standard basis of $\mathbb{R}^n$. Let $A = [y_1, \ldots, y_n]$ and $A^* = [z_1, \ldots, z_n]$ be $n \times n$ matrices. Show directly (using the standard coordinates) that $\mathcal{R}(A)^\perp = \mathcal{R}(A^*)$.
(3) In the situation of Problem (2), how are $\mathcal{R}(A)$, $\mathcal{R}(A^*)$, W, and $W^\perp$ related?

III. Transversal Matroids

Remember the bicircular matroid of a graph, $B(G)$? (Look it up in the book.)

(1) Show that $B(G)$ is a transversal matroid by finding a transversal presentation of a special kind. Which transversal presentations naturally give bicircular matroids?

(2) Characterize the dual bicircular matroids, using the theory of § 2.4 for dual transversal matroids.