ADDITIONAL ERRATA AND COMMENTS ABOUT OXLEY, *MATROID THEORY*

`http://www.math.binghamton.edu/zaslav/580.F04/my-errata.ps`

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NEW marks most errata that are new since 2001. **VERY NEW** since 2008.

Page 1: *Sets of positive numbers:* $\mathbb{Z}_+$ would be better than $\mathbb{Z}^+$, which should be reserved for the additive group of integers. This is by analogy with $\mathbb{R}^+, \mathbb{Q}^+$ which in certain areas of higher math are actually used for (so should be reserved for) the additive groups of real numbers, rationals, analogously to $\mathbb{R}^*, \mathbb{Q}^*$ for the multiplicative groups.

Page 3: Why not use the familiar notation F^r for the r-dimensional vector space instead of $V(r, F)$? **NEW**

Page 27: *Prop. 1.3.10:* Also, $\mathcal{D}''$ should not be empty. **VERY NEW**

Page 35: *Running head:* The 1.5 should be 1.4.

Page 35: *Problem 12:* The correct hypothesis is that $C_j \not\subseteq \bigcup_{i\neq j} C_i$.

Page 44: *Fig. 1.17* The line segment 6–8 should cross in front of 3–10, not the reverse.

Page 46: *1.6, par. 2, line 7:* "all j in J <u>and the e_j are distinct</u>." **NEW**

Page 57: *line 8:* It appears that "by 1.7.4" should read "by the chain condition". (Since $\bigvee X \vee y \geq \bigvee X$ and equal height imply equality.)

Page 57: The standard symbol for the partition lattice is Π_n. $\mathcal{P}_n$ is rare, at best.

Page 58: *proof of 1.7.8* There is no need for different cases. The second-case argument covers both cases.

Page 72: *corank:* This is an unfortunate choice of definition. (Not due to Oxley.) Not only is it common in matroid theory to define $\operatorname{corank}(S) = r(M) - r(S)$, but the latter definition is not confined to matroids but is also used in the more general theory of ranked posets (because of order duality). We will always use the definition: $\operatorname{cr}(S) := \operatorname{corank}(S) = r(M) - r(S)$. We will call r^* the *dual rank* function. **NEW**

Page 73: *Clutter:* The definition should state that $\mathcal{C} \neq \varnothing$. If not, then 2.1.12 is incorrect when $\mathcal{A} = \varnothing$ and 2.1.18 needs an additional axiom

$$\text{(H0)} \qquad \mathcal{H} \neq \varnothing,$$

and other corrections might be needed elsewhere. **NEW**

Page 78: *line −4:* In Oxley's erratum for this, a better replacement would be <u>on a known set</u>. **NEW**

Page 78: Definition of fundamental cocircuit: **NEW** I think the terminology is mistaken. One should call it "the fundamental cocircuit of e with respect to the basis B" with the notation $C^*(e, B)$. This is for consistency with graph theory, where the concept originated. One always speaks of "fundamental cocircuit [or cocycle] with respect to" a spanning tree, not a cotree. I'm not aware of any variation in this.

Besides that, I consider it preferable to define $C^*(\ )$ by (i) and replace (i) by "Show that $C^*_M(e, B) = C_{M^*}(e, E - B)$. This is also for consistency with graph theory; besides, it seems reasonable to give $C^*(\ )$ an independent meaning.

Page 85: line -12: "vector space" (no plural).

Page 85: The link between matroid duality and vector space orthogonality is much more than is stated in the book (Theorem 2.2.8 and Proposition 2.2.23), as we have seen in class. (Oxley chose not to go into greater detail about this, obviously because he couldn't do everything.) Here are some theorems that make good exercises. (The first is about representations but not duality.)

- *Theorem 2.2.A.* Given matrices A, A' over any field, with n columns. If they have the same row space, $\mathcal{R}(A') = \mathcal{R}(A)$, then they have the same column matroid: $M[A] = M[A']$.
- *Theorem 2.2.D.* Given matrices A, A^* over any field, with n columns. If $\mathcal{R}(A^*) = \mathcal{R}(A)^\perp$, then $M[A^*] = M^*[A]$.
- *Theorem 2.2.G (Whitney's Orthogonality Theorem).* In the Euclidean vector space $\mathbb{R}^n$ (with dot product), let $b_1, \ldots, b_n$ be an orthonormal basis and let W be a subspace. Let y_i be the orthogonal projection of b_i onto W and let z_i be its orthogonal projection onto $W^\perp$. Let $M = M[y_1\ \ldots\ y_n]$. Then $M[z_1\ \ldots\ z_n] = M^*$.

Page 91: There is a minor confusion of terminology here. Strictly speaking, a particular embedding of a graph in the plane is called a *plane graph* but is not a graph (i.e., not an abstract graph). A *planar graph* is an abstract graph that *can be* embedded in the plane; whether it is or is not embedded at the moment is irrelevant. Every plane graph has a unique planar dual (as discussed in class) which is a plane graph, not a planar graph. A planar graph gets a dual graph by being embedded in the plane and then dualized; there can be more than one dual, from choosing different embeddings.

Page 91: Plane graph: Also, an edge and vertex do not intersect in the interior of the edge. **NEW**

Page 91: Probably by an oversight, Oxley neglected to state the basic theorem:

Theorem 2.3.A. If G is a planar graph and G^* is a dual of G, then $M(G^*) = M^*(G)$.

Page 96: "Linked to" might be "linked onto" to distinguish it better from "linked into". **NEW**

Page 104: Terminology for contraction: Also called the contraction "of M by T" or "of M from T".

Page 124: γ: It is simpler to define the relation directly and forget about $\gamma(e)$. The definition is: $e\gamma f$ if $e = f$ or there is a circuit containing both e and f. **NEW**

Page 125: Definitions of connectedness and disconnectedness: I think one should say M is *connected* if it has exactly one component and *disconnected* if it has more than one component. Then the empty matroid is neither connected nor disconnected. This is how Tutte handles it for graphs in his book. However, one could define M to be disconnected if it doesn't have exactly one component. The main thing is that the empty matroid is *not connected*. This is clearly correct and intended by Oxley. **NEW**

Page 125: Proposition 4.1.4: "if and only if <u>$E(M)$ is nonempty and</u>, for every pair". **NEW**

Page 125: line 14, end: Eliminate space before ";".

Page 125: def. of "block": This is not the usual definition in graph theory, although usage varies. The usual definition is: a 2-connected graph (this ignores loops). When we need to distinguish the two types, we might call Oxley's an *edge block* and the usual one a *vertex block*. (There is nothing wrong with Oxley's definition at all. One should just be aware of the other meaning.)

Page 138: line -6: The -1 should be -1.

Page 145: line 7: This is an unfortunate definition. The natural terminology, that will never confuse anyone and scarcely needs to be defined, is: *planar matroid* for a matroid of rank 3 and *planar-graphic matroid* for the matroid of a planar graph. These are the generally used terms as well. In the course we will always say "planar" and "planar-graphic" as defined here, not as in the book.

Page 149: line 15, etc.: Instead of "generalized cycle" I suggest the term *necklace (of graphs)*. (The "parts" might then be called "beads".) One would speak of a "necklace of blocks" if the parts are blocks. **NEW** (in part).

Page 150: Proof of Lemma 5.3.5: Date added: December 1, 2004. I believe I have a simpler proof.

First, we observe that G_j's being a block $\implies M(G_j)$ is connected $\implies M(H_j)$ is connected $\implies H_j$ is a block. [As in the existing proof.]

Next, we note that in G there are only two kinds of cycles: a *local* cycle is contained in some G_j, and a *global* cycle contains an edge of every G_i. Therefore, the same is true in H.

Next we treat the case $k = 2$. We define $E_u \subseteq E(G_1)$ as in the book. We note that $\theta(E_u)$ is a bond, and therefore $H_1 - \theta(E_u)$ has the two components H_1', H_1. We also note that every global cycle contains an edge of E_u. Suppose H_1' has two or more contact vertices with H_2. Then there is a cycle in $H_1' \cup H_2$, which is not local but does not contain an edge of $\theta(E_u)$. This is impossible. Therefore there are exactly two contact vertices of H_1 with H_2, so we have a necklace of blocks, each with at least 3 vertices.

Finally, we treat the case $k > 2$. This uses the notion of the block-cutpoint graph of a graph and the theorem that the block-cutpoint graph is a forest, with one component for each component of the original graph. Since H_{-j} has at least two vertices in common with H_j and both graphs are connected, there is a cycle C that contains an edge of each. Thus C is global. Thus, $C \cap H_{-j}$ is a path in H_{-j} that contains an edge of every H_i, $i \neq j$. This is possible only if the block-cutpoint forest of H_{-j} is a path and P has endpoints that belong to the end blocks of that path and are not cutpoints of H_{-j}. It follows that H is a necklace of blocks $H_1, \ldots, H_k$ except that possibly H_{-j} has more than two contact vertices with H_j. However, in that case one of the end blocks has two (or more) contact vertices with H_j, and then there is a cycle contained in their union that is not local but not global; that is impossible. Thus, H is a necklace of blocks $H_1, \dots, H_k$, as desired.

Page 152: *line 17:* I think the $\theta(H_i)$ and $\theta(H_{-i})$ should not have θ's.

Page 164: We will not make a distinction between "projective geometry" and "projective space". I believe most writers, if pressed, would use the opposite convention to Oxley's ("geometry" for the axiomatic definition, "space" for the construction from coordinates), but most likely they wouldn't care. For consistency with most writings on projective geometry and matroids, we will call an abstract system (P, L, ι) as defined at the bottom of page 164 either a *projective geometry* or *projective space* (synonymously). The projective geometry $PG(n, F)$ over a field F will be called an *F-coordinatizable projective geometry (space)* or simply a *coordinatizable projective geometry (space)*. (These are also called, depending on the exact hypotheses, "pappian" and "desarguesian projective geometries" in the technical literature of projective geometry, for reasons that we need not go into.)

Page 164: *Abstract projective space:* The definition admits any simple matroid of rank at most 2, with the exception of the 2-point line $U_{2,2}$.

Page 165: *Theorem 6.1.1* The statement must say "dimension > 2" and "integer $n > 2$". The conclusion is false, not only for $n = 2$, but also for $n = 1$. Any simple matroid of rank 2 consisting of $q+1$ points where $q < 2$ or q is not a prime power is a counterexample. (Oxley's errata have this correction.)

Page 170: *Prop. 6.1.10:* The proof actually shows that the Vamos matroid is not a submatroid (therefore not a minor) of any projective geometry.

Page 175: *lines 23–24:* By the above remark on 6.1.10, the Vamos matroid is such a matroid.

Page 184: *line -6:* "is" should be "are" **NEW**

Page 185: *Definition of vector representation:* This is a good place to introduce a more fundamental definition. A *vector representation of M over F* is a mapping $f : E(M) \to F^m$ (for some $m \geq 0$) such that a set $S \subseteq E(M)$ is independent in M if and only if $f(S)$ is independent in F^m. We count multiplicity; thus if $f(x) = f(y)$, then $\{f(x), f(y)\}$ is dependent in F^m. (That is, $f(S)$ is a multiset of vectors.) The matrix representations previously defined are merely a particular way to present the mapping f; for some purposes a very valuable way, as we have seen.

A more general definition replaces F^m by any abstract vector space over F. This is perfectly valid but not especially necessary.

Page 185: *Equivalence of representations:* I believe the notion of equivalence in the book, which I shall call *matroid equivalence*, is not the best. A better one is that of projective equivalence. We call two representations of M over F *projectively equivalent* if one is obtained from the other by a combination of 6.3.1–6 (with no exception in dimension 2). (This definition is based on the fundamental paper of Brylawski and Lucas [1]. Welsh [3, p. 281] defines an equivalence that is projective equivalence without the field automorphisms.)

A *projective automorphism* of $\mathrm{PG}(n, F)$ is the projection of an invertible semilinear transformation T of F^{n+1}. That is, it is the effect of T on the simplification $\mathrm{PG}(n, F)$ of $M(F^{n+1})$. A *matroid automorphism* is any permutation that preserves the matroid structure (equivalently, it preserves collinearity). The fundamental theorem of projective geometry tells us, among other things, that every matroid automorphism of $\mathrm{PG}(d, F)$ is a projective automorphism, *provided that* $d \geq 2$. That is, for projective dimension 2 or more, matroid and projective automorphisms are the same. *If* $d = 1$*, this is not true.* A projective line $\mathrm{PG}(1, F)$, whose matroid is $U_{2,q+1}$ where $q = |F|$ (see Exercise 6.1.3), has as matroid automorphisms the full symmetric group of its points, but a projective automorphism is determined by its values on any 3 points (as I showed in class for the 3 particular points $[1, 0], [1, 1], [0, 1]$). There are more matroid automorphisms than projective automorphisms. (We can say that the projective structure of a projective line is not entirely determined by the matroid structure.)

Restating the two definitions of equivalence: *Oxley* calls two representations of M in a projective space $\mathrm{PG}(n, F)$ equivalent if one equals the other followed by a *matroid* automorphism of $\mathrm{PG}(n, F)$. They are *projectively* equivalent only if the matroid automorphism is a *projective* automorphism.

I will give some reasons for thinking that the correct automorphisms are the projective ones. (I might summarize as follows: Oxley thinks of $\mathrm{PG}(n, F)$ as a matroid, but I think of it as a coordinatized projective geometry, which therefore has more structure than just its matroid. I am following the tradition of projective geometry. It was not clear to me for a long time that the projective-geometry notion is superior, but I have finally concluded, after dealing with various representation problems, that it is, and that we are missing something important about representation of matroids if we ignore the projective structure on $\mathrm{PG}(n, F)$.)

There are several reasons for preferring projective to matroid equivalence. The simplest is that it eliminates special exceptions for (or exclusions of) rank 2 or dimension 2 in many results, e.g., Proposition 6.3.13 and (less obviously) Theorem 6.3.10. The more important reason is that it has good consistency properties.

To explain this, consider representing a matroid M of rank 2 (let's say, $U_{2,n}$ with $n > 3$) over a field F. (Assume F is large enough that a representation exists. That is, $|F| \geq n - 1$.) We could represent M in $\mathrm{PG}(1, F)$. Then, all representations of M are matroid equivalent. Now think of the $\mathrm{PG}(1, F)$ as a line in the plane $\mathrm{PG}(2, F)$. Suddenly, not all representations of M in that line are matroid equivalent. The reason is that not all matroid automorphisms of the line can be extended to matroid automorphisms of the plane. In fact, the automorphisms that can be extended are precisely the projective automorphisms of the line. This seems to me to be a serious difficulty. Oxley gets around it by requiring representations to be in projective spaces of rank equal to $r(M)$, but that is not such a good idea: it is artificial; worse, it is incompatible with some constructions, such as the lift construction (which we haven't studied).

(If you like, you can replace the $\mathrm{PG}(1, F)$ and $\mathrm{PG}(2, F)$ in the previous paragraph by F^2 and F^3. The same difficulty appears.)

Thus, if we want the property of unique representability over F to be independent of which dimension of projective or vector space we use for the representation, we have to prefer projective equivalence over matroid equivalence.

Page 186: line 6: "same size" means same dimension. (That is, the dimension of the vector space.)

Page 186: line −6: Properly speaking, what Oxley is defining is an *invertible* semilinear transformation.

Page 186: line -5*:* "there is" should be "there are".

Page 189: line 10: "there is" should be "there are".

Page 190 ff.: This is supposed to be read "D-sharp". (The correct musical symbol is $\sharp$, not # (pound (weight) sign.)

Page 193: Prop. 6.4.5: It might be desirable to elucidate this proposition by the following lemmas, whose proofs are obvious. If the rows of a matrix A are labelled $1, \ldots, r$ and the columns are labelled $1, \ldots, n$ (corresponding to the matroid points $e_1, \ldots, e_n$), then $A[I, J]$ denotes the minor of A consisting of the elements in rows $i \in I$ and columns $j \in J$.

Lemma 1. *Let A be an $r \times n$ matrix representing a matroid M of rank r. Let $S \subseteq [n]$ with $|S| = r$. Then S corresponds to a basis of M if and only if* $\det A\big[[r], S\big] \neq 0$.

Lemma 2. *Let $A = [\, I \mid D\,]$ in Lemma 1. Then S corresponds to a basis of M if and only if $D\big[[r] \setminus S, S \setminus [r]\big] \neq 0$.*

Page 209: Lemma 6.6.2: The statement is incorrect. It is not sufficient that M be binary; it is necessary to assume that $[\, I_r \mid D_1\,]$ itself is a GF(2)-representation matrix of M.

Furthermore, it should be made clear that the entries of D_1 are in $\{0, \pm 1\}$ when viewed in characteristic 0 (e.g., over $\mathbb{R}$) and the entries of D_2 will then be in $\{0, \pm 1\}$ when viewed in characteristic 0. (This is necessary because $\operatorname{char} F$ may not be 0.)

Another comment: it seems only necessary that $[\, I_r \mid D_1\,]$ represent M over some field of characteristic not 2 and some field of characteristic 2.

Oxley has a correction for this lemma, but I don't think it captures everything.

Page 210: $\mathbb{R}$ should be F (lines 2, 12, -8). **NEW**

Page 210: line 21: I would rather say that (i) $\Rightarrow$ (ii) due to the obvious fact that a totally unimodular matrix representing M over $\mathbb{R}$ (or $\mathbb{Q}$) also represents M over any field.

Page 212: line -7*:* "over F, <u>and also over GF(2) because, viewing D and $D^\sharp$ over $\mathbb{R}$, $D \equiv D^\sharp \mod 2$.</u>"

Page 213: para. 2: Oxley's Errata pages give a correction for this paragraph. The following is my commentary on the original paragraph, which supplements Oxley's correction.

The pivoting is done over F, not $\mathbb{R}$ since we don't know the matrix represents M over $\mathbb{R}$. We have to know from the proof of Lemma 6.6.2 that the entries in D_2 are still all in $\{0, \pm 1\}$ even when viewed in characteristic 0, because the only way a pivot can produce a different value of an entry is to produce a 2, which is ruled out by the fact that it would be $0 \mod 2$ but not in F, therefore contradicting the assumption that M is represented by $[\, I_r \mid D_1\,]$ over both GF(2) and F. The conclusion is that the determinant of D' is in $\{0, \pm 1\}$ in characteristic 0. Thus, $[\, I_r \mid D_1\,]$ is totally unimodular and represents M in characteristic 0.

Page 213: line 21: Just a thought: Perhaps, strictly speaking, these results are matroid *analogs*, not generalizations, of Kuratowski's theorem. Kuratowski's theorem does not follow from Tutte's results, except by use of Whitney's 2-isomorphism theorem, which is rather complicated.

Page 214: #4 line 1: A is surely intended to be an integral matrix. (This might be considered implicit but I felt some confusion about interpreting the question.) **NEW**

Page 230: top equation: This is called the "modular law". Contrast with the semimodular law.

Page 248, 287: Grammar: "2-sum" (meaning the operation of performing a 2-sum) should be "2-summation" or "2-summing". **NEW**

Page 258: #4: It would be reasonable to call $\mathcal{N}$ a *modular filter of sets.* (I have a discussion of this in "Biased graphs. II", page 59 (top), where I take the dual point of view, of a modular ideal of sets. The modular ideal corresponding to a modular filter $\mathcal{N}$ is the set of complements of members of $\mathcal{N}$. If $\mathcal{N}$ is in M, then the modular ideal is in M^*. The modular filter is derived from copoints of M; the modular ideal is derived from circuits in M^*.) **NEW**

Page 262–4: Exposition: Much of this proof would benefit from the use of corank, $\mathrm{cr}(X) = r(M) - r(X)$. For instance, (iii) with $Y = E$ says that $\mathrm{cr}_1(X) \geq \mathrm{cr}_2(X)$. Lemma 7.3.7 concerns the case of equality.

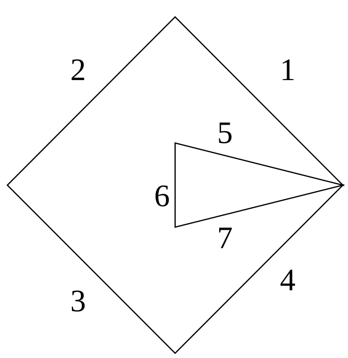
I suggest that the best way to think of Lemma 7.3.7 is in terms of what is actually a corollary, let's call it *Lemma 7.3.7′*: If $X \in \mathcal{L}(M_2)$ and $\mathrm{cr}_1(X) = \mathrm{cr}_2(X)$, then $X \in \mathcal{L}(M_1)$ and $[X, E]_1 = [X, E]_2$. **NEW**

Page 278: 8.2.1 This is really Whitney's extension of Menger's theorem. Menger's theorem is 8.2.1 for nonadjacent vertex pairs only. (Each is stronger in one of the two directions.) See some graph theory books (not Tutte's). (Any book has Menger's theorem; not many have Whitney's.) **NEW**

Page 278: Line after 8.2.2: "n-connectedness" should be "n-connectedness". **NEW**

Page 278 ff.: Despite Tutte's prestige, I think the adjective should be "verticial", not "vertical". Reasons: (1) It's confusing since it doesn't contrast with "horizontal". (2) I think it's not etymologically justified. Compare "matrix/matricial" and "simplex/simplicial" with "vertex/verticial". **NEW**

Page 284: Definition of equivalent planar embeddings: The definition by having the same sets of edges bounding each face is wrong. A and B ought to be inequivalent. The definition should say that the *sequence* of bounding edges is the same, up to cyclic permutation. It's not so clear how to treat C and D. Maybe also reversing the sequence should be considered equivalent; this would allow reflections to be equivalent, and I think but I'm not sure it won't create improper equivalences. **NEW**



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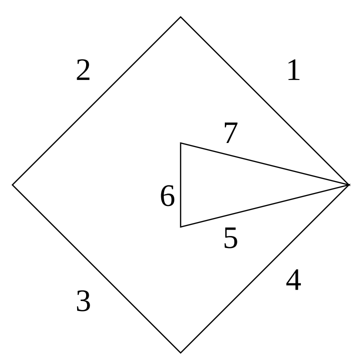

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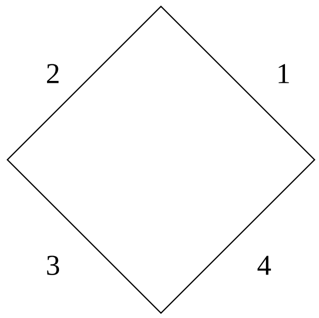

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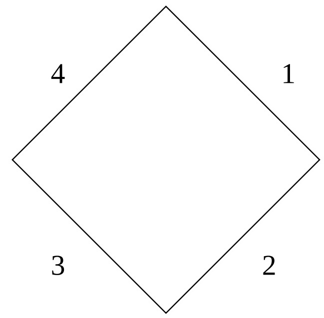

D

Page 284: *Line 16* Add comma after "5.3". **NEW**

Page 284: *Line 19* Add comma after "Evidently". **NEW**

Page 284: *Line 20* The comma may be omitted. **NEW**

Page 289: *Corollary 8.3.4* "minors of itself"; "operations of direct summation and 2-summation". **NEW**

Page 289: It is worth mentioning that $M_1 = M|((X_1 \cup C)/([X_2 \cap C] \backslash p_1)$ and a similar formula for M_2. (These are good exercises.) **NEW**

Replacing p_1 and p_2 by a new element p, then M is the 2-sum $M_1 \oplus_p M_2$ (where the meaning of $\oplus_p$ should be clear). **NEW**

Page 309: *Ex. 2:* I think this isn't exactly right (but I don't recall why). Am I wrong? **NEW**

Page 314: *Ex. 2(i):* In the matrix, the A_4 should be I_4.

Page 314: *Ex. 3(ii):* "all non-zero chains in N".

Page 314: *Chain-groups:* It is a pity that the only careful and modern treatment of chain-groups is that in Tutte's textbook [2, Ch. VIII], and that this treatment avoids mentioning matroids. It would be very desirable to have a development of the main points in a matroid book, specifically, a section in this book. (Tutte's book is "modern" in that it has the notion of a primitive chain-group, which generalizes chain-groups over integral domains and clarifies some of the basic ideas.)

Page 393: *Corollary 12.2.17:* The M and $n(M)$ here are different from those in 12.2.16.

Also, I think that 12.2.17 can be restated in a simpler way: "Let M be a transversal matroid. There there is an integer $N(M)$ such that M is representable over every field having at least $N(M)$ elements."

Page 465: *Title (and elsewhere?):* The correct title is "Unimodality conjectures". The conjectures are not unimodal, they concern unimodality. **NEW**

Page 523: *Index of Notation:* Add: G^*, 91.

Add: D_n, 52. **NEW**

Page 524: *Index of Notation:* The operation of complementary dualization of $\mathcal{S} \subseteq 2^E$ is omitted from the list. Its definition, $\mathcal{S}' = \{E \setminus S : S \in \mathcal{S}\}$, is implicit on page 69, proof of Corollary 2.1.5.

$\mathcal{R}(\mathcal{A})$ should be $\mathcal{R}(A)$. **NEW**

Page 528: Index: Add: lattice of divisors of an integer, 52.

Page 529: Index: modular short circuit axiom, p. *234*.

Page 530: Index: Add: regular matroid, 83.

Page 532: Index: vertex–edge incidence matrix: p. *4*. **NEW**

References

[1] T.H. Brylawski and D. Lucas, Uniquely representable combinatorial geometries. In: *Colloquio Internazionale sulle Teorie Combinatorie* (Proc., Rome, 1973), Vol. I, pp. 83–104. Atti dei Convegni Lincei, 17. Accademia Nazionale dei Lincei, Rome, 1976. MR 55 #113 (book). Zbl. 392.51007.

[2] W.T. Tutte, *Graph theory.* Encyc. Math. Appl., 21. Addison-Wesley, Reading, Mass., 1984. MR 87c:05001. Zbl. 554.05001. Reprinted by Cambridge University Press, Cambridge, 2001. MR 2001j:05002. Zbl. 964.05001.

[3] D.J.A. Welsh, *Matroid Theory.* Academic Press, London, 1976. MR 55 #148. Zbl. 343.05002.