Perfect Graph Theory | Instructor: Padraic Bartlett

Lecture 3: Chordal Graphs

Week 1 | *Mathcamp 2011*

When we defined perfect graphs in our last lecture, the idea was that they would be graphs with "easy-to-determine" chromatic numbers. Has this been true so far?

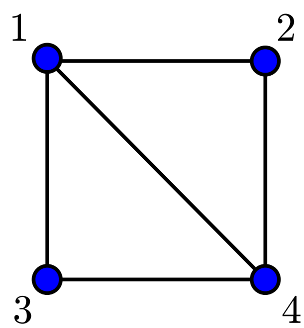
For the three classes of graphs we've shown to be perfect: yes! Complete graphs have trivial-to-find chromatic numbers, bipartite graphs also have easily-found chromatic numbers, and so do their line graphs (the chromatic number of the line graph $L(G)$ of a bipartite graph G is $\Delta(G)$, which we proved in our last lecture.)

Motivated by this success, our lecture today will invert this process: first, we'll try to find a family of graphs that (if they were perfect) would have an easily-calculated chromatic number, and then we'll see if this means they're actually perfect after all!

1 Perfect Elimination Orderings

Definition. In a graph G, a vertex v is called **simplicial** if and only if the subgraph of G induced by the vertex set $\{v\} \cup N(v)$ is a complete graph.

For example, in the graph below, vertex 3 is simplicial, while vertex 4 is not:



A graph G on n vertices is said to have a **perfect elimination ordering** if and only if there is an ordering $\{v_1, \dots v_n\}$ of G's vertices, such that each v_i is simplicial in the subgraph induced by the vertices $\{v_1, \dots v_i\}$. As an example, the graph above has a perfect elimination ordering, witnessed by the ordering $(2, 1, 3, 4)$ of its vertices.

Why do we mention this definition? Well: if a graph G admits a perfect elimination ordering, then we have a really fast way to find its clique number: just look at the n different cliques

- $(\{v_n\} \cup N(v_n)) \cap \{v_1, \dots v_n\}$,
- $(\{v_{n-1}\} \cup N(v_{n-1})) \cap \{v_1, \dots v_{n-1}\}$,
- ...

- $\{v_1\}$.

If H is an induced subgraph corresponding to a maximum-size clique in our graph, and $v_k \in V(H)$ is the vertex in H with the largest subscript value in our ordering, then by definition $H = (v_k \cup N(v_k)) \cap \{v_1, \ldots v_k\}$; therefore, H comes up in our list! So, to find the largest clique, we just have to check n different graphs. This stands in sharp contrast to the normal situation for graphs, where finding $\omega(G)$ is a NP-complete problem.

In a very well-defined sense, then, we've shown that graphs that have perfect elimination ordering are graphs that would have really easy to find chromatic numbers (if they were perfect!) So: are they?

As it turns out: yes! We prove this here, in two propositions:

Proposition 1 *If G admits a perfect elimination ordering, so do any of its induced subgraphs.*

Proof. Let $\{v_1, \ldots v_n\}$ be G's perfect elimination ordering, $i_1 < i_2 < \ldots i_k$ be any subsequence of the sequence $\{1, 2, \ldots n\}$, and H the corresponding induced subgraph of G on $\{v_{i_1}, \ldots v_{i_k}\}$. By definition, we had that each of the graphs

- $(\{v_{i_k}\} \cup N(v_n)) \cap \{v_1, \ldots v_{i_k}\}$,
- $(\{v_{i_{k-1}}\} \cup N(v_{n-1})) \cap \{v_1, \ldots v_{i_{k-1}}\}$,
- $\ldots$
- $\{v_{i_1}\} \cup N(v_{i_1})) \cap \{v_1, \ldots v_{i_1}\}$

were cliques in G; therefore, by restricting to H, we have that all of the sets

- $(\{v_{i_k}\} \cup N(v_n)) \cap \{v_{i_1}, \ldots v_{i_k}\}$,
- $(\{v_{i_{k-1}}\} \cup N(v_{n-1})) \cap \{v_{i_1}, \ldots v_{i_{k-1}}\}$,
- $\ldots$
- $\{v_{i_1}\}$

are still cliques. Therefore, this induced subgraph H still admits a perfect elimination ordering.

Proposition 2 *If G admits a perfect elimination ordering, G is perfect.*

Proof. By our above proposition, it suffices to just show that $\chi(G) = \omega(G)$ for any graph G with a simplicial elimination ordering. We proceed by induction on the number of vertices in G. If $|V(G)| = 1$, G is trivially perfect, as it's K_1.

Assume now that $V(G) = n > 1$, for some n, and let $\{v_1, \ldots v_n\}$ be the perfect elimination ordering of G's vertices that we're given. Look at the graph $G \setminus \{v_n\}$, formed by deleting v_n from G. By our proposition, $G \setminus \{v_n\}$ still admits a simplicial elimination ordering. Therefore, we can apply our inductive hypothesis to see that $G \setminus \{v_n\}$ is perfect: i.e. that $\chi(G \setminus \{v_n\}) = \omega(G \setminus \{v_n\})$.

For brevity's sake, define $k = \omega(G \setminus \{v_n\})$. In G itself, by definition, we know that the collection of vertices $v_n \cup N(v_n)$ induces a clique as a subgraph: therefore, we know that $N(v_n)$ itself induces a clique, and therefore that $\deg(v_n) = |N(v_n)| \leq \omega(G \setminus \{v_n\}) = k$. So v_n has less than k neighbors.

Suppose that $\deg(v_n) < k$. Then, given any k-coloring of $G \setminus \{v_n\}$, we can extend it to a coloring of G by just letting v_n be whatever color in $\{1, \ldots k\}$ doesn't show up in its neighbors. This means that $\chi(G) = k = \omega(G \setminus \{v_n\}) \leq \omega(G)$, and therefore that G is perfect.

Conversely, assume that $\deg(v_n) = k$. Then $v_n \cup N(v_n)$ forms a clique of size $k+1$, so $\omega(G) \geq k+1$. Finally, because $\chi(G \setminus \{v_n\}) = k$, we can extend any k-coloring of $G \setminus \{v_n\}$ to a $k+1$-coloring of G by painting v_n the color $k+1$; this shows that $\chi(G) \leq k+1 \leq \omega(G)$, and therefore (again) that G is perfect.

Excellent! The only somewhat unsatisfying part of this new family of graphs is that their property – this perfect elimination ordering – is a kind of ponderous thing, and not quite as obviously easy to check as (say) being bipartite, or being the line graph of a bipartite graph. One of the other motivations we had for defining perfect graphs was our hope that it would lead us to a "nice" characterizing property, similar to the one we had for bipartite graphs; does one exist for these "perfect elimination ordering" graphs?

As it turns out, yes!

2 Chordal Graphs

Definition. A graph G is said to contain a **chordless cycle** if and only if it has some induced subgraph isomorphic to a cycle C_t, for $t \geq 4$. If a graph does not contain any chordless cycles, it is called **chordal**.

Definition. For any two vertices $x, y \in G$ such that $\{x, y\} \notin E(G)$, a $x - y$ **separator** is a set $S \subset V(G)$ such that the graph $G \setminus S$ has at least two disjoint connected components, one of which contains x and another of which contains y.

Theorem 3 *For a graph G on n vertices, the following conditions are equivalent:*

1. *G has a perfect elimination ordering.*
2. *G is chordal.*
3. *If H is any induced subgraph of G and S is a vertex separator of H of minimal size, S's vertices induce a clique.*

Proof. ($1 \Rightarrow 2$:) Let C be any cycle in G of length at least 4. Take our perfect elimination ordering of G, and start deleting vertices according to this ordering until you get to an element c in C. When you delete this element in C, we know that its neighbors in C have to induce a clique: therefore, there is a "chord" (i.e. edge) between two elements in C, and therefore the induced subgraph on the vertices in C is not a cycle.

($2 \Rightarrow 3$:) Any induced subgraph of a chordal graph is chordal, because any cycle in G has a chord in it, which will be preserved in any induced subgraphs containing that cycle.

So it suffices to prove that if G is chordal, any minimal $x - y$ separator S will induce a clique.

To do this: let S be a minimal $x-y$ separator in G, and let A_x, A_y be the two connected components of G that contain x and y, respectively. Suppose that u, v are a pair of vertices in S; then, because S is minimal, there are edges from both u and v to the two components A_x, A_y (otherwise, we wouldn't have needed them to separate A_x from A_y. Let P_x be the shortest path from u to v in A_x, and P_y be the shortes path from u to v in A_y; because both of these paths have length ≥ 2, their union is a cycle of length ≥ 4. Because G is assumed to be chordal, there must be a chord in this cycle; because there are no direct edges from A_x to A_y (because they're distinct connected components when we cut along S) nor any other edges from u, v to these components (because we picked shortest-possible paths), the only possible chord can be if $\{u, v\}$ itself is an edge! Because this holds for every pair of vertices $u, v \in S$, we have that there is an edge between every pair of vertices in S: i.e. S induces a clique.

($3 \Rightarrow 1$:) We proceed by induction on n, the number of vertices in G. For $n = 1$ this is trivial; so we assume that our claim holds for all graphs on $\leq n - 1$ vertices, and seek to prove it for graphs on n vertices. If G is a clique, we are trivially done, as any ordering of G's vertices gives a perfect elimination ordering. Otherwise, there are a pair of vertices x, y such that $\{x, y\}$ is not an edge in $E(G)$. Let S be a minimal $x - y$ separator, and A_x, A_y the components of $G \setminus S$ containing x and y, respectively. By the inductive hypothesis, the component A_x has a perfect elimination ordering; in specific, there is a vertex $u \in A_x$ such that $\{u\} \cup N(u)$ is a clique in the induced subgraph on A_x's vertices. Therefore, because this vertex has no edges to any other connected components (by definition), and because S's edges form a clique, we know that $\{u\} \cup N(u)$ forms a clique in our original graph, G.

Let $v_n = u$ in our perfect elimination ordering, and delete u from G; this leaves us with a graph on $n - 1$ vertices with our desired property, which by induction has a perfect elimination ordering $\{v_1, \ldots v_{n-1}\}$. Combining this ordering with our v_n then gives us a perfect elimination ordering of G.

Perfect elimination graphs, therefore, are chordal – a remarkably elegant classification!