DUALITY
MATH 580A, SPRING 2013

I. Abstract Duality

(1) Duality in terms of closures. (Assume all sets are finite.)
   (a) Given a matroid M, for all $S \subseteq E$ and $x \in E \setminus S$, define $T = E \setminus S \setminus x$. Then either $x \in \text{cl}(S)$ or $x \in \text{cl}^*(T)$, but not both. (This is Oxley's Exercise 2.2.5.)
   (b) Given two closure operators on a set E, such that for all $S \subseteq E$ and $x \in E \setminus S$, either $x \in \text{cl}(S)$ or $x \in \text{cl}^*(E \setminus S \setminus x)$, but not both. Then cl and cl^* are the closure operators of a dual pair of matroids. (This is due to Crapo. It's bigger than (a) because we don't assume the closures are matroid closures.)

(2) Find a self-dual axiom system based on circuits and cocircuits and the property

$$|C \cap C^*| \neq 1 \quad \forall C \in \mathcal{C},\ C^* \in \mathcal{C}^*.$$

II. Duality of Vector Representations

Theorems of concern here are:

- Theorem 2.2.A,B = Exercise 2.2.6(a,b).
- Theorem 2.2.8.
- *Theorem 2.2.W (Whitney's Orthogonality Theorem).* In the Euclidean vector space $\mathbb{R}^n$ (with dot product), let $b_1, \ldots, b_n$ be a basis and let W be a subspace. Let y_i be the orthogonal projection of b_i onto W and let z_i be its orthogonal projection onto $W^\perp$. Let M be the vector matroid of $y_1, \ldots, y_n$. Then the vector matroid of $z_1, \ldots, z_n$ is M^*.

Problems:

(1) A theorem of linear algebra states: Suppose you have a subspace $W \leq \mathbb{R}^n$ with a basis $\alpha_1, \ldots, \alpha_r$ and you form the matrix A whose rows are the vectors α_i, $i = 1, \ldots, r$. Then the orthogonal projection onto W of any vector $x \in \mathbb{R}^n$ is given by the formula $\text{proj}_W x = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}Ax$. Find out how to prove this formula, either by looking it up or by working it out yourself.

(2) Make the assumptions of Theorem 2.2.W with the addition that $\{b_1, b_2, \ldots, b_n\}$ is the standard basis of $\mathbb{R}^n$. Let $A = [y_1, \ldots, y_n]$ and $A^* = [z_1, \ldots, z_n]$ be $n \times n$ matrices. Show directly (using the standard coordinates) that $\mathcal{R}(A)^\perp = \mathcal{R}(A^*)$.

(3) In the situation of Problem (2), how are $\mathcal{R}(A)$, $\mathcal{R}(A^*)$, W, and $W^\perp$ related?

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III. Transversal Matroids

Remember the bicircular matroid of a graph, $\mathrm{BG}(G)$?

(1) Show that $\mathrm{BG}(G)$ is a transversal matroid by finding a natural transversal presentation. Which transversal presentations naturally give bicircular matroids?

(2) Characterize the dual bicircular matroids, using the theory of § 2.4 for dual transversal matroids.

IV. Spikes

The graph $2C_n$ is a circle $C_n = e_1 e_2 \cdots e_n$ with each e_i doubled by a parallel edge f_i. (When needed, I name the vertices $v_1, v_2, \ldots, v_n = v_0$ with $V(e_i) = V(f_i) = \{v_{i-1}, v_i\}$.)

We saw in class that a tippy spike is the extended (or complete) lift matroid $L_0(2C_n, \mathcal{B})$ of a biased graph $(2C_n, \mathcal{B})$. Here $E(L_0) = E(2C_n) \cup \{d_0\}$. The tipless spike is the lift matroid, $L = L_0 \setminus d_0$. I suggest you use that representation of spikes in the following.

(1) Let $I \subseteq [n]$ and define

$$\begin{cases} s_I(e_i) = f_i,\ s_I(f_i) = e_i & \text{if } i \in I, \\ s_I(e_i) = e_i,\ s_I(f_i) = f_i & \text{if } i \notin I. \end{cases}$$

Question: When is s_I an isomorphism of L with its dual, $L^\perp$? For instance, we know it is for $I = \varnothing$ when $\mathcal{B} = \varnothing$. The answer obviously depends on $\mathcal{B}$.