NOTES ON LINE GRAPHS
VERY PRELIMINARY DRAFT
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Contents

1. Introduction

/{intro} /

These notes are the true representation and explanation of line graphs, signed graphs, directed graphs, and geometry. All other representations are false and untrue and not as weird; or at least, different.

Why do we care about line graphs? The line graph of a graph tells us what the relations are between edges. Whitney long ago asked: do we need vertices? Can we tell everything just from edges? In other words, can a graph be reconstructed from its line graph? The answer: Almost always. This is one reason we like line graphs.

Why do we care about graph eigenvalues? There are some specific theorems, but the general idea is that it can be hard to calculate important facts about graphs. Eigenvalues

Conceived around 1980. Written 1 December 2006 – ?.

can tell you some of those facts, and eigenvalues may not be so hard to find. This leads one to look into eigenvalues, and it turned out that line graphs had a remarkable eigenvalue property: the least eigenvalue is -2, or greater. Furthermore, several interesting line graphs are completely determined by their spectrum. Maybe all graphs with the eigenvalue lower bound of -2 are line graphs? That turned out not to be true, because of Alan J. Hoffman's generalized line graphs, but it was finally discovered by Cameron, Goethals, Seidel, and Shult that aside from those, there hardly are any other graphs with -2 as a lower bound. This work is deservedly famous.

Meanwhile, signed graphs appeared on the scene, sometimes incognito, sometimes in disguise. If there are signed graphs, don't they have eigenvalues, just like us graphs? Do they not have line graphs, just like us? And do not our methods work as well for them? Maybe so! Read on!

2. Weird kinds of graph

/{graphs} /

2.1. **Graphs.** We have a more general notion of a graph than is usual. There are three kinds of edges. A *link* is an edge with two distinct endpoints and a *loop* is an edge with two coinciding endpoints. A *half-edge* has one endpoint. Thus, for instance, in calculating the degree of a vertex, a loop counts twice but a half-edge only once.

So far this is not remarkable, but in addition, we have to think of an edge as composed of one or two *ends* (short for "edge ends"). We think of an edge e as made up of two ends if a link or loop, one end if a half-edge. Each end belongs to exactly one edge and has exactly one vertex (its *endpoint*); the endpoints of an edge are precisely the endpoints of the ends of the edge. Then, a *graph* Γ is a triple (V, E, H), where V is the vertex set, E is the edge set, and H is the set of ends.

All our graphs are finite. They may have multiple edges; if there are no loops, no half-edges, and no parallel edges, then Γ is called *simple*. Two ends, or two edges, are *adjacent* if they have a common endpoint. The number of vertices is $n = |V|$. The set of links and loops is E_*. A *circle* is a connected 2-regular subgraph, or its edge set. The class of circles is $\mathcal{C}$.

The *adjacency matrix* $A(\Gamma)$ is a $V \times V$ matrix whose off-diagonal entry a_{vw} equals the number of links between v and w, and whose on-diagonal entry is

$$a_{vv} = 2(\text{number of loops}) + (\text{number of half-edges}) \text{ at } v.$$

An *incidence matrix* of Γ is a $V \times E$ matrix $D(\Gamma)$ whose (v, e) entry is

- 0 if v is not incident with e, or if e is a loop at v,
- ± 1 if v is incident with e and e is not a loop, with the rule that if e is a link its endpoints have opposite sign.

A well known property of the adjacency and incidence matrices is that

$$/\{\text{E:gincidadj}\}\ /D(\Gamma)D(\Gamma)^{\mathrm{T}} = \Delta(\Gamma) - A(\Gamma), \tag{2.1}$$

where the *degree matrix* $\Delta(\Gamma)$ is the diagonal matrix whose (v, v) element is the degree of vertex v. (Recall that the degree is the number of edges plus the number of loops incident with v, so that a loop is counted twice.)

One often sees in the literature a different incidence matrix, where all -1's are replaced by 1's and (if loops are admitted at all, which is rare) the entry in position (v, e) for a loop

e at v is 2 instead of 0. We call this the *unoriented incidence matrix* of Γ and write $D_0(\Gamma)$ for it. It satisfies the formula

$$/\{\text{E:unorincidadj}\}\ /D_0(\Gamma)D_0(\Gamma)^{\mathrm{T}} = \Delta(\Gamma) + A(\Gamma). \tag{2.2}$$

Let us define a symmetric relation of *incidence* on $V \cup E \cup H$ by saying an edge and its ends are incident, and also an edge or end and its endpoint(s) are incident. An *isomorphism* of graphs Γ_1 and Γ_2 is a function $\theta : V_1 \cup E_1 \cup H_1 \to V_2 \cup E_2 \cup H_2$ which is a bijection $V_1 \to V_2$, $E_1 \to E_2$, and $H_1 \to H_2$ and which preserves incidence of edges, ends, and vertices.

2.2. **Signed graphs.** A *signed graph* $\Sigma = (\Gamma, \sigma)$ is a graph together with a *signature* $\sigma : E_* \to \{+1, -1\}$, the two-element group. (A half-edge does not receive a sign.) Two especially important kinds of signed graph are $+\Gamma$, in which every edge is positive, and $-\Gamma$, in which every edge has negative sign (in both cases excepting half-edges since they have no sign). We say Σ is *simply signed* if it has no positive loops, no half-edges, and no multiple edges with the same sign; but it may have parallel links of opposite sign. A *signed simple graph* is a simple graph Γ with signs. A signed simple graph is simply signed, but a simply signed graph need not be simple, as it can have a *negative digon*, that is, a pair of parallel edges of opposite sign.

Reducing a signed graph means deleting negative digons until none remain; a signed graph without negative digons is called *reduced*. A signed simple graph is reduced. A simply signed graph may not be reduced, but its reduction is a signed simple graph.

The *adjacency matrix* $A(\Sigma)$ is the $V \times V$ matrix in which the value of a_{vw} for distinct vertices v, w is the number of positive edges vw less the number of negative edges vw, while

$$a_{vv} = \begin{cases} 2\,(\text{number of positive loops}) \\ -\,2\,(\text{number of negative loops}) \\ +\,(\text{number of half-edges}) \end{cases} \quad \text{incident with } v.$$

An *incidence matrix* of Σ is a $V \times E$ matrix $D(\Sigma)$ whose (v, e) entry is

- 0 if v is not incident with e, or if e is a positive loop at v,
- ± 1 if v is incident with e and e is not a loop, with the rule that the endpoints of e have opposite sign if e is positive and identical sign if e is negative,
- ± 2 if e is a negative loop at v.

Observe that an incidence matrix of a signed graph, like that of an unsigned graph, is not unique since it remains an incidence matrix if any column is negated. It is easy to verify that

$$/\{\text{E:sgincidadj}\}\ /D(\Sigma)D(\Sigma)^{\mathrm{T}} = \Delta(\Gamma) - A(\Sigma). \tag{2.3}$$

[VERIFY FOR LOOPS, HALF-EDGES.] In this respect we see that an unsigned graph behaves just like the all-positive signed graph $+\Gamma$; (2.1) is a special case of (2.3). On the other hand, we may take the unique incidence matrix of $-\Gamma$ that has no negative entries; this is the unoriented incidence matrix $D_0(\Gamma)$, and since $A(-\Gamma) = -A(\Gamma)$, (2.2) is another special case of (2.3).

Each circle C (and indeed any walk that does not have a half-edge) has a sign $\sigma(C)$ obtained by multiplying the signs of its edges; $\mathcal{B} = \mathcal{B}(\Sigma)$ is the class of positive circles. A subgraph or edge set in Σ is called *balanced* if it has no half-edges and every circle is positive; it is *antibalanced* if $-\Sigma$ is balanced.

An *isomorphism* of signed graphs Σ_1 and Σ_2 is an isomorphism $\theta : \Gamma_1 \xrightarrow{\sim} \Gamma_2$ of underlying graphs that preserves edge signs; that is, $\sigma(e) = \sigma(\theta(e))$ for each edge $e \in E_1$.

Signed graphs and the notions of balance and antibalance were introduced by Harary in [5, 6]. The incidence matrix appears in [17].

2.3. **Bidirected graphs.** In a bidirected graph every end has an orientation; thus, a loop or link has two directions, one at each end. Formally, a *bidirected graph* B is a pair (Γ, β) consisting of a graph and an end signature $\beta : H \to \{+1, -1\}$, which we call a *bidirection* of Γ. We think of β as orienting the ends so that a positive end is directed towards its vertex and a negative end is directed away from its vertex. The *incidence matrix* $D(\mathrm{B})$ is the $V \times E$ matrix in which the (v, e) entry is the sum, over all ends ε of e whose endpoint is e, of the signs $\beta(\varepsilon)$. That is,

- 0 if v is not incident with e, or if e is a positive loop at v,
- ± 1 if v is incident with e and e is not a loop, with the rule that the endpoints of e have opposite sign if e is positive and identical sign if e is negative,
- ± 2 if e is a negative loop at v.

Observe that the incidence matrix of a bidirected graph is unique.

Bidirected edges fall into three types. First, there are the edges whose ends have opposite signs (such edges must be links and loops); their two directions point consistently along the edge, so they give the edge a single direction. Then there are edges whose ends are both positive; we call them *extraverted* because their directions point outwards, toward the incident vertices. Last are the edges whose ends are negative; they are *introverted* edges because their directions point into the edge. A half-edge has to be introverted or extraverted. A bidirected graph is *negatively homogeneous* if every edge is introverted or every edge is extraverted.

We associate to each bidirected graph B the signed graph $\Sigma(\mathrm{B})$ in which a link or loop e with ends ε_1 and ε_2 has sign

$$[\{\text{E:bisign}\}]\sigma(e) = -\beta(\varepsilon_1)\beta(\varepsilon_2). \tag{2.4}$$

Obviously, the incidence matrix of B is an incidence matrix of $\Sigma(\mathrm{B})$. Given a signed graph Σ, we call any bidirected graph B for which $\Sigma(\mathrm{B}) = \Sigma$ an *orientation of* Σ; that is, a bidirected graph is an orientation of a signed graph. We see that the nonuniqueness of the incidence matrix of a signed graph is due to the fact that Σ has several orientations, just as with ordinary graphs (which in this way again behave like all-positive signed graphs.) As the positive edges of B are directed, they form a directed graph; conversely, any directed graph can be viewed as a bidirected graph all of whose edges are positive. On the other hand, the negative edges have no intrinsic direction.

The *adjacency matrix* of a bidirected graph is defined to be the adjacency matrix of its signed graph.

Bidirected graphs B_1 and B_2 are called *isomorphic* if there is an isomorphism $\theta : \Gamma_1 \xrightarrow{\sim} \Gamma_2$ of underlying graphs that preserves the bidirection; i.e., $\beta_1(\varepsilon) = \beta_2(\theta(\varepsilon))$ for each end $\varepsilon \in H_1$.

Bidirected graphs are due to Edmonds [3], and later, independently, by Zítek [21] and Zelinka under the name “polarized graph”. Zaslavsky [19] observed that they are oriented signed graphs.

2.4. **Polar graphs.** A *polar graph* is a graph together with a bipartition of the edge ends incident to each vertex. That is, the ends incident with v are divided into two subsets, say M_v and N_v (possibly empty).

If we define $\beta : H \to \{+1, -1\}$ in such a way that, at every vertex, β is constant on M_v andon N_v but take opposite values on each, then we get a bidirected graph. Conversely, if we take a bidirected graph and forget the values of β but remember which ends at v have the same sign, we get a polar graph.

Polar graphs Π_1 and Π_2 are *isomorphic* if there is an isomorphism of underlying graphs that preserves the bipartitions at corresponding vertices.

Polar graphs were introduced by Zelinka in [20] *et al.*

2.5. **Switching.** *Switching* a vertex set $W \subseteq V$ in a signed graph Σ means reversing the signs of all edges with one endpoint in W and the other in its complement W^c. We write Σ^W for the switched graph. A convenient way to represent this is by means of a *switching function* $\eta : V \to \{+1, -1\}$. We define the *switching of* Σ *by* η to be the signed graph $\Sigma^\eta = (\Gamma, \sigma^\eta)$ whose signature is given by the rule

$$\sigma^\eta(e) = \eta(v)\sigma(e)\eta(w) \tag{2.5}$$

if the endpoints of e are v and w. (The rule naturally does not apply to a half-edge as it has no sign.) When η is chosen to be negative on W and positive on W^c, then $\Sigma^\eta = \Sigma^W$.

The essential property of switching a signed graph is that it leaves the sign of every circle invariant. Thus, $\mathcal{B}(\Sigma^\eta) = \mathcal{B}(\Sigma)$.

Lemma 2.1 ([17])**.** [{L:switch}] *It is possible to switch a signed graph so that any desired balanced subgraph has only positive edge signs.*

Switching a bidirected graph B is similar. The directions of ends with endpoints in W are reversed, while the other directions remain the same. The result is denoted by B^W. If we have a switching function, then

$$\beta^\eta(\varepsilon) = \eta(v)\beta(\varepsilon), \tag{2.6}$$

when ε is an edge end whose vertex is v. Note that switching a bidirected graph by -1 reverses the directions of all ends, but switching a signed graph by -1 has no effect on the signs. An edge signature is not subtle enough to detect all switching, but a bidirection is.

Lemma 2.2. [{L:biswitch}] *It is possible to switch a bidirected graph so that any specified forest has a predetermined orientation.*

Proof. Switch so that the signs on the forest are as desired. Then the orientation of each tree is either the desired one, or its opposite. If the latter, switch all the vertices of the tree. □

Call two signed, or bidirected, graphs *switching equivalent* if one can be switched to become the other, and *switching isomorphic* if one can be switched to be isomorphic to the other. An equivalence class, called a *switching class*, is denoted by $[\Sigma]$ or $[\mathrm{B}]$. A switching class $[\mathrm{B}]$ of bidirected graphs, called a *bidirected switching class*, is obviously equivalent to the polar graph associated with B (but it is not the same; a polar graph is not a class of equivalent bidirected graphs). A switching class $[\Sigma]$ of signed graphs, called a *signed switching class*, is determined by its family $\mathcal{B}(\Sigma)$ of positive circles [17]:

Lemma 2.3. [{L:switchingequiv}] *Two signed graphs are switching equivalent if and only if they have the same underlying graph and the same positive circles.*

Therefore, a switching class is equivalent to what is called a *sign-biased graph*, namely, the pair $(\Gamma, \mathcal{B}(\Sigma))$. One can characterize the possible sets $\mathcal{B}(\Sigma)$ as the intersection of $\mathcal{C}(\Gamma)$ with some subspace of the binary cycle space of Γ [16].

In matrix terms switching negates the rows of the incidence matrix that correspond to switched vertices (or, in terms of the switching function, vertices on which it is negative). Thus, it negates the corresponding rows and columns of the adjacency matrix. What it does not change is the eigenvalues of $A(\Sigma)$ (or their multiplicities). Therefore, although the adjacency matrix of a switching class is only determined within conjugation by a diagonal sign matrix, the eigenvalues of a switching class are well defined.

Switching of graphs was introduced by Seidel in [9, 12]. Switching of signed graphs, from [17] *inter alia*, is a generalization, suggested by the fact that Seidel's adjacency matrix of a simple graph Γ equals the adjacency matrix of a signed complete graph K_Γ, and Seidel switching of Γ is the same as signed switching of K_Γ. Switching of bidirected graphs is the obvious extension of signed switching.

3. LINE GRAPHS

/{linegraphs} /

3.1. **The definition.** /{lgdef} / We want to define line graphs of signed graphs, bidirected graphs, and polar graphs, but first we have to define the line graph of a plain old graph, since that is the basis of the other definitions and our graphs are more complicated than in the usual treatments of line graphs.

Graphs. /{lg} / The line graph of a graph $\Gamma = (V, E, H)$ is denoted by $L(\Gamma)$. The vertex set is $V(L(\Gamma)) = E$. The edge set is

$$E(L(\Gamma)) = \big\{\{\varepsilon, \varepsilon'\} : \ \varepsilon, \varepsilon' \text{ are adjacent ends in } H\big\}.$$

Thus, $L(\Gamma)$ has an edge for every pair of adjacent ends in Γ. The ends in $L(\Gamma)$ are the ordered pairs $(\varepsilon, \varepsilon')$ such that $\{\varepsilon, \varepsilon'\}$ is a line-graph edge. The ends $(\varepsilon, \varepsilon')$ incident to a line-graph vertex e are those such that ε is incident to e as an edge of Γ. A loop in Γ has the somewhat peculiar effect that, because its ends ε_1 and ε_2 are adjacent, they create a loop $\{\varepsilon_1, \varepsilon_2\}$ in the line graph whose ends are $(\varepsilon_1, \varepsilon_2)$ and $(\varepsilon_2, \varepsilon_1)$.

A line graph L has two kinds of cliques. A *vertex clique* in L is the set of all L-vertices that, as Γ-edges, are incident with a single vertex in Γ. A *triangular clique* is the set of all L vertices that are the edges of a subgraph of Γ induced by three mutually adjacent vertices.

It also has two important kinds of circles. A *vertex triangle* is a triangle contained in a vertex clique. A *derived circle* is the line graph of a circle in Γ. These circles generate the binary cycle space.

Hoffman's generalized line graphs. Hoffman introduced a generalization of line graphs. To explain it we define the *cocktail party graph* $CP(m)$; this is K_{2m} with the edges of a perfect matching deleted. Now, choose a vertex weighting $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^V_{\geq 0}$. In $L(\Gamma)$, the edges of Γ incident to each vertex v_i of Γ form a clique, call it the ith vertex clique. The *generalized line graph* $L(\Gamma; \mathbf{m})$ is obtained by taking the disjoint union of $L(\Gamma)$ with the n cocktail party graphs $CP(m_i)$ for $1 \leq i \leq n$ and joining every vertex of the $CP(m_i)$ to every vertex of the ith vertex clique in $L(\Gamma)$.

Hoffman's construction is really a line graph of a signed graph. The signed graph is obtained by adding to each vertex v_i of $-\Gamma$ exactly m_i negative digons. Each digon has one vertex v_i and the other a new vertex of degree 2. When we take the reduced line graph of this expanded signed graph, the negative digons yield a cocktail party graph adjacent to all the vertices that arise from edges of Γ incident to v_i. This is clear if one chooses an orientation in which every vertex of Γ is a sink (all ends directed into the vertex); then every edge of $\Lambda'(\Gamma; \mathbf{m})$ is negative except the two parallel edges derived from an added digon, and these two cancel in the reduction. (This observation about digons has been made independently elsewhere, but in an *ad hoc* manner, without signs. Our contribution is to show that it is part of a regular process of eliminating edges whose signs cancel.)

Bidirected and directed graphs. /{bilg} / The line graph of a bidirected graph $\mathrm{B} = (\Gamma, \beta)$, written $\vec{\Lambda}(\mathrm{B})$, is the line graph $L(\Gamma)$ with bidirection $-\beta$, the negative of the end signature in B. (We sometimes like to call this a *line bidirection* of Γ.) The graphical interpretation is that the direction of an end remains the same but its character reverses: if it was directed into its vertex in B, it is directed away from its vertex in the line graph. See Figure 3.1/{*F:bilinegraph*}/.

FIGURE 3.1. A picture of a bidirected graph and its line graph.
[{F:bilinegraph} *]*

The line graph of a digraph deserves a closer look. A digraph is an oriented all-positive signed graph. The edges of its line graph are either positive, hence directed edges, or negative, hence introverted or extraverted directed edges. If we keep just the positive part of the line graph, what we have is the Harary–Norman line digraph [7]. If we keep only the negative part and reinterpret it as a signed graph, extraverted edges being positive and introverted ones being negative, then we have the signed line graph of a digraph of Muracchini and Ghirlanda [10]. *[VERIFY THE SIGN CHOICE.]* (It is rare but not unique in the literature for a "signed graph" to be really an all-negative bidirected graph; the difference may often be told from the incidence matrix.)

Signed graphs. *[*{sgrlg} *]* Given a signed graph $\Sigma = (\Gamma, \sigma)$, we define its polar line graph $\vec{\Lambda}(\Sigma)$ by taking any orientation of Σ, that is, a bidirection β such that $\Sigma(\Gamma, \beta) = \Sigma$, and finding its line graph, $\vec{\Lambda}(\Gamma, \beta)$. Reorienting an edge in Σ is the same as switching the corresponding vertex of $\vec{\Lambda}(\Gamma, \beta)$; therefore, the polar line graph $\vec{\Lambda}(\Sigma)$ of a signed graph is a switching class of bidirected graphs, or equivalently, a polar graph.

The bipartition of ends at a vertex e of the line graph is easy to describe. If e was negative or a half-edge, there is only one class of edge ends at e as a line-graph vertex; if e was positive with ends ε_1 and ε_2, there are two classes, one containing the ends of the form $\{\varepsilon_1, \varepsilon'\}$ and the other the ends of the form $\{\varepsilon_2, \varepsilon'\}$.

In practice, however, we are interested mostly in the signed switching class of the line graph. We call this the *line graph of the signed graph* and write it as $\Lambda(\Sigma)$.

Polar graphs. *[*{polarlg} *]* Taking the line graph of a polar graph is complementary to taking a line graph of a signed graph. Think of a polar graph as a switching class $[\mathrm{B}]$ of bidirected graphs. Switching a vertex of B means reorienting the edges of its vertex clique. Thus, the line graph of a polar graph Π is a class of orientations of a signed graph Σ, but it may not be the class of all orientations. We can treat $\Lambda(\Pi)$ as a signed graph if we show that the line-graph signature σ determines the switching class of the bidirection, because then the orientation chosen for Σ does not matter; either it is a line bidirection of the original polar graph $\Lambda(\Pi)$, or it is not a line bidirection at all. We will do this soon in Proposition 3.1*[{P:polarlg}]*.

Proposition 3.1. [{P:polarlg}] *Let* $[\mathrm{B}_1]$ *and* $[\mathrm{B}_2]$ *be switching classes of bidirected graphs that have the same underlying graph* Γ*, and let* $\Sigma_i = \Lambda(\mathrm{B}_i)$*. Then* $\Sigma_1 = \Sigma_2$ *if and only if* $[\mathrm{B}_1] = [\mathrm{B}_2]$*.*

Proof. We may assume Γ is connected.

If $[\mathrm{B}_1] = [\mathrm{B}_2]$, switch B_2 so it equals B_1. Switching does not change the signs of the line graph, so Σ_2 is unchanged, but now $\Sigma_2 = \Lambda(\mathrm{B}_1)$.

Conversely, by assumption $\Lambda(\mathrm{B}_1)$ and $\Lambda(\mathrm{B}_2)$ have the same signs. Therefore, the latter is a reorientation of the former. Switch B_2 so that its orientations agree with B_1 on some maximal forest F. This does not change Σ_2. We wish to prove that now $\mathrm{B}_2 = \mathrm{B}_1$. We look at a vertex clique, $E_v \subset E(\Gamma)$, whose vertices are the edges $e_1, \ldots, e_k$ incident with v in Γ; choose the labels so $e_1 \in F$. Let ε_j be the end of e_j that is incident with v. (There are two such ends if e_i is a loop.) The orientation of ε_j, $j > 1$, equals $-\beta_i(\varepsilon_1)\sigma(\varepsilon_1, \varepsilon_j)$, where σ is the sign in the line graph; this is the same for $i = 1, 2$. Therefore, after switching B_2 does equal B_1. □

Switching classes and sign bias. Combining switching and reorientation, we conclude that the line graph of a switching class of signed graphs is a well defined switching class of signed graphs.

Put slightly differently, the reduced and unreduced line graphs of a sign-biased graph are sign-biased graphs. That is the viewpoint of [18]. We take from there the description of which circles are positive in the line graph. We begin with two lemmas about signed graphs.

Lemma 3.2. [{L:vertexcliquesign}] *In a line graph of a signed graph, a vertex clique is antibalanced.*

Proof. It is easy to see that every triangle in a vertex clique is negative. □

Lemma 3.3. [{L:derivedsign}] *In a line graph of a signed graph, the sign of a derived circle $\Lambda(C)$ equals the sign of C.*

Proof. Let l be the length of C. The sign of C equals the product, over all edges $e_i \in C$, of $-\beta(\varepsilon_{i1})\beta(\varepsilon_{i2})$, where ε_{i1} and ε_{i2} are the ends of e; thus it is $(-1)^l$ times the product of all $\beta(\varepsilon_{ij})$. The sign of $\Lambda(C)$ is the product of $(-1)^l$ and the product of all $-\beta(\varepsilon_{ij})$. Since there are evenly many ends, the sign of $\Lambda(C)$ equals that of C. □

Proposition 3.4. [{P:lgbalance}] *Let Λ be the line graph of a signed switching class* $[\Sigma]$. *A circle C of Λ has sign $(-1)^{t+q}$ if it is the set sum of t vertex triangles, q negative derived circles, and p positive derived circles.*

Proof. The proposition follows from four facts. First, the vertex triangles and derived circles span the binary cycle space. Second, the sign of a circle is a homomorphism from the additive group of binary cycles (i.e., set sums of circles) to the sign group. Third, every vertex triangle is negative. Fourth, Lemma 3.3*[{L:derivedsign}]*. □

3.2. **Reduced line graphs.** *[*{reduced} *]* The signed line graph $\Lambda(\mathrm{B})$ may have parallel edges with opposite sign. Such pairs contribute 0 to the adjacency matrix because their contributions cancel. (This applies to loops as well as links.) Thus, we define the *reduced line graph* $\Lambda'(\mathrm{B})$ to be the result of removing from $\Lambda(\mathrm{B})$ as many pairs of oppositely signed parallel edges as possible. Its adjacency matrix is the same as that of $\Lambda(\mathrm{B})$.

The reduced line graph, as a bidirected graph, is not usually unique, not even up to bidirected isomorphism, because, for instance, there may be several negative edges to pair with a given positive edge, and they may not be oriented similarly. However, when we look at the signed line graph or line graphs of switching classes, reduction is unique up to vertex-fixing isomorphisms. Thus, reduction belongs most properly with signed line graphs and line graphs of switching classes, rather than bidirected graphs.

The work of Vijakumar *et al.* is based on signed switching classes, though they do not describe the signed graphs they study as line graphs and all their graphs are reduced line graphs. *[MAKE SURE OF THAT!]*

3.3. **Natural classes; graphic line signed graphs.** *[*{natural}*]* We have seen three closed systems under taking line graphs: bidirected graphs, all-negative signed graphs (or antibalanced signed switching classes), and signed switching classes (or sign-biased graphs). The familiar one of these is all-negative signed graphs, which correspond to ordinary line graphs. We should add to the list the class of negatively homogeneous signed graphs—taking the line graph converts a homogeneously extraverted bidirected graph to a homogeneously introverted one and vice versa—because this class is the technical means by which all-negative signed graphs give rise to all-negative line graphs.

We want to find out how a line graph can be antibalanced, or balanced.

Proposition 3.5. [{P:antibalancedlg}] *Suppose Σ is connected. The line graph $\Lambda(\Sigma)$ is antibalanced if and only if Σ is antibalanced except for half-edges. The reduced line graph $\Lambda'(\Sigma)$ is antibalanced if and only if Σ, without its half edges, is an antibalanced signature of a generalized line graph.*

Proof. [NEEDS PROOF!] □

3.4. **Reduced line graphs that are graphs.** *[*{lggraphs}*]*

A reduced line graph may be treated as an unsigned graph if all its edges have the same sign. For a switching class, that means the class is balanced (so the signs can be made all positive) or antibalanced (so the signs can be taken to be all negative).

All negative signs. We saw that the right sign to get the ordinary line graph of the underlying graph as the line graph of a signed graph is the negative one; that is, one takes $\Sigma = -\Gamma$. Then, if we orient every edge to be extraverted (by taking $\beta \equiv +1$), the line graph has every edge introverted, hence negative, and the line graph $\Lambda(-\Gamma)$ equals $-L(\Gamma)$. Since negating the graph negates the eigenvalues, we get graphs with least eigenvalue not less than -2. The classical question, answered in [1], is to find all other graphs with that eigenvalue bound. In our terms that means all antibalanced signed graphs whose eigenvalues are not greater than 2. Our theory explain why Hoffman's generalized line graphs are such graphs.

All positive signs. The opposite sign is also possible. If we begin with an all-positive graph, its line graph is rarely all positive; in fact, it is rarely balanced. The existence of a vertex triangle prevents that. The real question, though, is whether we end up with a balanced line graph. Such line graphs can be treated as all positive and thus as unsigned graphs. They are graphs that have all eigenvalues ≤ 2. These were also found in [1].

Those that are line graphs of signed graphs. Our contribution to the discussion is the observation that most of the graphs of the two kinds just mentioned, with eigenvalues ≥ -2 or ≤ 2, that are not line graphs of ordinary graphs are actually line graphs of signed graphs.

[MORE: SAY WHAT THEY ARE.]

4. Eigenvalues

[{evalues}] It is well known that, when Γ is simple, the eigenvalues of the adjacency matrix of its line graph are bounded below by -2. The proof is that, since every diagonal element of $D_0(\Gamma)^{\mathrm{T}} D_0(\Gamma)$ equals 2 and $A(L(\Gamma))+2I = D_0(\Gamma)^{\mathrm{T}} D_0(\Gamma)$, the matrix $A(L(\Gamma))+2I$ is positive semidefinite.

Hoffman's generalized line graphs also have least eigenvalue -2 or greater [8]. The real reason is that they are reduced line graphs of signed graphs, as we show below.

4.1. Signed graphs.

A consequence of our previous discussion is that for the study of eigenvalues we may look at signed switching classes. However, for calculations, as for instance in the next lemma, we have to fix an orientation in order to have well defined incidence and adjacency matrices.

Lemma 4.1. [{L:lgformula}] *Let Σ be a signed graph and let Λ be any signed graph in $\Lambda([\Sigma])$, the line graph of its switching class. Then*

$$[\{\text{E:lgformula}\}\ ]A(\Lambda) = 2I - D(\Sigma)^{\mathrm{T}}\, D(\Sigma).$$

Proof. We examine m_{ef}, the (e, f) entry of $M = D(\Sigma)^{\mathrm{T}}\, D(\Sigma)$. m_{ef} is the dot product of the columns of D belonging to e and f. To do the calculations we fix the orientation B whose incidence matrix is D.

If e and f are nonadjacent, the entry is 0.

If they are parallel links, let them have ends ε_1 and ε_2 of e and ε'_1 and ε'_2 of f, with ε_i adjacent to ε'_i. Then

$$m_{ef} = \beta(\varepsilon_1)\beta(\varepsilon'_1) + \beta(\varepsilon_2)\beta(\varepsilon'_2) = \beta(\varepsilon_1)\beta(\varepsilon'_1)\big[1 + \sigma(e)\sigma(f)\big].$$

This equals 0 if ef is a negative digon and $2\beta(\varepsilon_1)\beta(\varepsilon'_1)$ if it is positive. In the latter case, we have two edges from e to f in the line graph, which we may write as $\varepsilon_1\varepsilon'_1$ and $\varepsilon_2\varepsilon'_2$. Their signs both equal $-\beta(\varepsilon_1)\beta(\varepsilon'_1)$, so their contribution to a_{ef} makes $a_{ef} = -m_{ef}$, as we wished.

We assume henceforth that e and f meet at only one vertex, v. If neither is a loop, let ε and ε' be their ends at v. The dot product equals $\beta(\varepsilon)\beta(\varepsilon')$. The sign of the edge equals $-\beta(\varepsilon)\beta(\varepsilon')$. Thus, $m_{ef} = -a_{ef}$, unless $e = f$. In that case, $m_{ef} = 1 = a_{ef}$ so $a_{ef} = 2 - m_{ef}$, as desired.

If e is a loop, let its ends be ε_1 and ε_2. Suppose f is not a loop and its end at v is ε'. Then the line graph has two parallel edges from e to f. The dot product is

$$m_{ef} = \big[\beta(\varepsilon_1) + \beta(\varepsilon_2)\big]\beta(\varepsilon') = \big[1 - \sigma(e)\big]\beta(\varepsilon_1)\beta(\varepsilon').$$

This is 0 if e is positive; and in that case the two parallel edges have opposite sign so $a_{ef} = 0$. If e is negative, then the two parallel edges have the same sign so $a_{ef} = -2\beta(\varepsilon_1)\beta(\varepsilon')$; while $m_{ef} = 2\beta(\varepsilon_1)\beta(\varepsilon') = -a_{ef}$.

Suppose both e and f are loops; let ε'_1 and ε'_2 be the ends of f. Now there are four ef edges in the line graph if $e \neq f$, or one loop if $e = f$. The incidence matrix has $\beta(\varepsilon_1) + \beta(\varepsilon_2)$ in position (v, e) and a similar expression in position (v, f), so

$$m_{ef} = \big[\beta(\varepsilon_1) + \beta(\varepsilon_2)\big]\big[\beta(\varepsilon'_1) + \beta(\varepsilon'_2)\big];$$

this is 0 if either e or f is positive, and otherwise it equals $4\beta(\varepsilon_1)\beta(\varepsilon'_1)$. When $e \neq f$, the four ef edges in the line graph have signs $-\beta(\varepsilon_i)\beta(\varepsilon'_j)$ for all combinations of $i, j = 1, 2$. The sum of these values is $-m_{ef}$. As for the case $e = f$, then the one loop has the same sign as e (by Lemma 3.3[{*L:derivedsign*}]), so the adjacency matrix has $a_{ee} = 2\sigma(e)$. The incidence matrix leads to the value $m_{ee} = 0$ if e is positive and 4 if it is negative, so $m_{ee} = 2 - 2\sigma(e)$, as we wanted. $\square$

The eigenvalue bound follows at once.

Theorem 4.2. [{T:lgevalues}] *The line graph of a bidirected graph has eigenvalues bounded above by* 2. $\square$

5. Line systems

/{lines} /

A *system of lines* is a set (finite unless we say otherwise) of lines through the origin of some $\mathbb{R}^m$. We are interested in systems of lines whose angles are among those that arise from signed graphs with eigenvalues at most 2.

Take two lines L_1, L_2 in a system $\mathcal{L}$. Define $\mathcal{L}(L_1, L_2)$ to be the set of lines of $\mathcal{L}$ that lie in the plane $\langle L_1, L_2 \rangle$ generated by L_1 and L_2. We say $\mathcal{L}$ is *star closed* if, for any two lines $L_1, L_2 \in \mathcal{L}$, the subsystem $\mathcal{L}(L_1, L_2)$ is closed under reflection in any line in it, or equivalently, under every rotation of $\langle L_1, L_2 \rangle$ that carries one line of the subsystem to another. The *star closure* of a system $\mathcal{L}$ is the smallest star-closed system of lines containing it. It is not obvious that the star closure is actually finite; this will be dealt with later.

Any finite set of nonzero vectors generates a system of lines in the obvious way.

A line system $\mathcal{L}$ is *decomposable* if it is the union of subsets $\mathcal{L}_1$ and $\mathcal{L}_2$ and every line of $\mathcal{L}_1$ is orthogonal to every line of $\mathcal{L}_2$.

5.1. **The angles of a signed graph with bounded eigenvalues.** /{angles} /

We start with a simply signed graph Σ of order m whose eigenvalues are not greater than 2. Being simply signed means that Σ has no multiple links of the same sign and no positive loops, and a vertex supports either one half-edge or one negative loop or neither. (One could also assume Σ is reduced, since reduction does not affect the adjacency matrix, but we need not do so.) Thus, the matrix $G := 2I - A(\Sigma)$ has diagonal entries equal to 1 for a vertex that supports a half-edge, 4 for a vertex supporting a negative loop, and 2 for any other vertex. Since G is positive definite, it is the Gram matrix of a set of m vectors $x_1, \ldots, x_m$, or in other words, $G = X^{\mathrm{T}} X$ where X is an $n \times m$ matrix for some value of $n \geq \operatorname{rank} A$. (The x_i are the columns of X.) Let $\mathcal{L}(\Sigma)$ be the system of lines in $\mathbb{R}^n$ generated by the columns of X; it is a multiset since some of the columns x_i might be collinear. The system lies in $\mathbb{R}^n$ and is isometric to every system $\mathcal{L}(\Sigma)$ in $\mathbb{R}^{\operatorname{rank} A}$; thus it is uniquely determined up to isometries except for the arbitrary choice of $n \geq \operatorname{rank} A$. Each line of $\mathcal{L}(\Sigma)$ corresponds to a vertex v_i of Σ; we denote it by $L_i = \langle x_i \rangle$.

The norm of x_i is 2 if v_i supports a negative loop, 1 if v_i supports a half-edge, and $\sqrt{2}$ if v_i supports neither. Let us say v_i, or x_i or L_i, has class respectively 0, 1, and 2 in each of these cases, write V_i for the set of vertices of class i, and classify the lines of $\mathcal{L}(\Sigma)$ similarly into $\mathcal{L})(\Sigma)$, etc.

Proposition 5.1. [{P:angles}] *Let Σ be a simply signed graph. The angles formed by distinct lines L_i, L_j of $\mathcal{L}(\Sigma)$ belong to the following list:*

(a) $90°$, *if v_i, v_j are nonadjacent,*
(b) $60°$, *if $v_i \in V_1$ and $v_j \in V_2$ are adjacent,*
(c) $45°$, *if $v_i n V_1$ and $v_j \in V_0 \cup V_2$ are adjacent.*

Proof. From $A(\Sigma)$, the inner product $x_i \cdot x_j$ is 0 in case (a). In case (b), the vectors have length $\sqrt{2}$ and their inner product equals ± 1. In case (c), they have lengths $\sqrt{2}$ and 1 and the inner product equals ± 1 so their angle has cosine $\pm 1/\sqrt{2}$; or they have lengths $\sqrt{2}$ and 2 and the inner product equals ± 2, so the angle also has cosine $\pm 1/\sqrt{2}$. ☐

From Proposition 5.1/{*P:angles*}/ we can write down a table of possible angles in $\mathcal{L}(\Sigma)$.

Norm	2	1	$\sqrt{2}$
Class	0	1	2
0	$0°, 90°$		
1	$45°, 90°$	$60°, 90°$	
2	$90°$	$45°, 90°$	$90°$

TABLE 5.1. The possible angles between lines of the three classes. The upper half of this symmetric table is omitted.

Define a *triple system of lines* as a system of lines with a tripartition $\mathfrak{L} = \mathfrak{L}_0 \cup \mathfrak{L}_1 \cup \mathfrak{L}_2$ into three pairwise disjoint subsystems such that the angles between lines of classes i and j are those allowed by Table 5.1*[{Tb:angles}]*. (A subset $\mathfrak{L}_i$ can be empty.) Thus, $\mathfrak{L}(\Sigma)$ is a triple system of lines.

5.2. **Root systems.** *[*{rs} *]*

If x is a nonzero vector in $\mathbb{R}^n$, let ρ_x denote reflection in the perpendicular hyperplane of x; that is, $\rho_x(y) = y - 2(x \cdot y)/(x \cdot x)$. A *root system* is a nonempty, finite set R of nonzero vectors in $\mathbb{R}^n$ with the following three properties:

(R1) For any vector $x \in R$, the only vectors in $R \cap \langle x \rangle$ are $\pm x$. *[*{E:rsmultiple} *]*
(R2) For any two vectors $x, y \in R$, the reflection $\rho_x(y) \in R$. *[*{E:rsreflect} *]*
(R3) If $x, y \in R$, then $2(x \cdot y)/(x \cdot x)$ is an integer. *[*{E:rsinteger} *]*

[VERIFY DEFINITION.] All root systems are known (there are many sources for this; one is [14, Chapter ??]). Suppose $R \subset \mathbb{R}^n$ and $R' \subset \mathbb{R}^{n'}$ are two root systems; then $R'' := (R \times \{0\}) \cup (\{0\} \times R'$ is a root system in $\mathbb{R}^{n+n'}$; such a root system R'' is called *reducible*. All root systems are constructed from indecomposable ones in this way, so it suffices to know all indecomposable root systems. To describe them, let $b_1, \ldots, b_n$ be the standard unit basis vectors of $\mathbb{R}^n$. The irreducible root systems are the *classical root systems*,

- $A_{n-1} := \{\pm(b_j - b_i) : 1 \leq i < j \leq n\} \subset \mathbb{R}^n$ for $n > 1$,
- $D_n := A_{n-1} \cup \{\pm(b_j + b_i) : 1 \leq i < j \leq n\} \subset \mathbb{R}^n$ for $n \geq 1$,
- $B_n := D_n \cup \{\pm b_i : 1 \leq i \leq n\} \subset \mathbb{R}^n$ for $n \geq 1$,
- $C_n := D_n \cup \{\pm 2b_i : 1 \leq i \leq n\} \subset \mathbb{R}^n$ for $n \geq 1$,

and the *exceptional root systems*,

- $E_8 := \{\} \subset \mathbb{R}^8$,
- $E_7 := \{\} \subset \mathbb{R}^7$,
- $E_6 := \{\} \subset \mathbb{R}^6$,
- $G_2 := \{\} \subset \mathbb{R}^2$,
- $F_4 := \{\} \subset \mathbb{R}^4$.

[VERIFY LIST.]

We see that A_{n-1} is the set of vectors representing all orientations of edges of the signed graph $+K_n$, D_n is the set of vectors representing $\pm K_n$, B_n represents $\pm K_n^\bullet$ (which is $\pm K_n$ with a half-edge at every vertex), and C_n represents $\pm K_n^\circ$ (that is, $\pm K_n$ with a negative loop at every vertex). The consequence is that, if $\mathfrak{L}$ is any subsystem of the line system generated by any of these four root systems, then $\mathfrak{L} = \mathfrak{L}'(\Sigma)$ for some simply signed graph

Σ. *[MAKE SURE ALL THESE ARE DEFINED. MOVE DEFS OF THE GRAPHS TO GRAPH SECTION.]*

Lemma 5.2. [{L:rsstar}] *The system of lines generated by any root system is star closed.*

Proof. This is obvious from (R2), since reflection of a line across another line L in a plane is the same as reflecting it across the perpendicular line $L^\perp$. □

We write $\mathcal{R}$ for the system of lines generated by a root system R, e.g., $\mathcal{A}_{n-1}$ is the line system generated by A_{n-1}. A *system of root lines* is any subset of a line system generated by a root system.

5.3. **Lines at** $60°$ **and** $90°$**.** *[{6090}]*

In this subsection we consider the theory for simply signed graphs that have no loops or half edges; that is (after reduction), for signed simple graphs.

A *broad system of lines* means a set of lines making angles of $60°$ and $90°$. It is therefore a triple line system in which $\mathcal{L}_0$ and $\mathcal{L})1$ are empty. The theory here is a simple extension of the classical theory of lines at angles $60°$ and $90°$ due to Cameron, Goethals, Seidel, and Shult [1]. ([4, Chapter 12] has a nice treatment.)

To each signed simple graph Σ of order m with eigenvalues not greater than 2 there is associated a broad system of lines. Conversely, given any broad system $\mathcal{L}$ of lines, there is a signed simple graph Σ. The vertex set $V = \{v_1, \ldots, v_m\}$ is in one-to-one correspondence with $\mathcal{L}'$. The signed graph is obtained by choosing one vector of length $\sqrt{2}$ in each line, forming their Gram matrix G (whose diagonal is 2), and defining Σ by $A(\Sigma) = 2I - G$. Since Σ is determined uniquely up to switching, there is a one-to-one correspondence between isometry classes of broad line systems and switching isomorphism classes of signed graphs with eigenvalues ≤ 2. (This construction does not work if the system is not broad.)

A broad system of lines is star closed if, whenever two of its lines, L_1 and L_2, make an angle of $60°$, the third line in the same plane that makes an angle of $60°$ with both, L_3, is also in the system.

Lemma 5.3 ([1])**.** [{T:starclosure6090}] *The star closure of a broad system of lines is a broad system in the same dimension.*

This means that, to classify all broad systems, it is sufficient to classify those that are star closed.

Lemma 5.4 ([1])**.** [{T:star6090}] *A star-closed, broad system of lines is either A_{n-1} or D_n or E_n.*

Since D_n represents the line graphs of $\pm K_n$, $A_{n-1} \subset D_n$, and $E_6 \subset E_7 \subset E_8$, we obtain the main theorem of [1] in its generalization to signed simple graphs:

Theorem 5.5. [{T:lg6090}] *A signed simple graph Σ with all eigenvalues not larger than 2 either is a line graph of a simply signed graph without loops or half-edges, or is a signed graph representable by E_8.*

An obvious corollary is that, if Σ has more vertices than the number of lines in E_8 (which is 128), or if its maximum valency is greater than that of the signed graph of E_8, then it has to be a line graph. More careful analysis leads to stronger conclusions. For instance, Chawathe and Vijayakumar found a list of ?? signed simple graphs of order at most 6 such that a signed simple graph is a line graph if and only if no induced subgraph belongs to the list [2].

We can develop the structure theory a little further without too much trouble. The question is which simply signed signed graphs are represented by broad systems of lines. Let us consider what star closure means for Σ. The two lines correspond to vectors x_1 and x_2 of norm $\sqrt{2}$ whose inner product is ± 1. That means the corresponding vertices of Σ are adjacent. By switching we can assume $x_1 \cdot x_2 = -1$. Then L_3 gives a vector x_3 which may be chosen so that $x_3 = -x_1 - x_2$. Then $x_3 \cdot x_1 = x_3 \cdot x_2 = -1$. Thus, the vertices v_1, v_2, v_3 support a negative triangle in Σ. Switching Σ, corresponding to choosing the negative of any of the vectors x_i, does not change the sign of the triangle. We conclude that Σ can contain only one negative triangle on each edge. Every edge is in a negative triangle if and only if $\mathcal{L}'$ is star closed.

This says nothing about positive triangles. *[MORE ON POSITIVE TRIANGLES??]*

[MORE ON WHAT STAR CLOSURE MEANS TO Σ?? (I.e., how the new vertex is connected to the existing graph.)]

[MORE ON CLASSIFICATION???]

5.4. **Lines at $45°$, $60°$, and $90°$.** *[*{456090}*]*

Now we allow any simply signed graph. We want analogs of the results of Cameron, Goethals, Seidel, and Shult; but now they have to apply to a triple line system rather than just a system with specified angles.

Lemma 5.6. [{L:linesrs}] *Let $\mathcal{L}$ be an indecomposable, star-closed triple system of lines. Then $\mathcal{L}$ is the set of lines generated by an irreducible root system other than G_2 and F_4*[OR SOMETHING LIKE THAT].

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