

**Graphs, Gain Graphs, and Geometry**  
**a.k.a.**  
**Signed Graphs and their Friends**

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# CHAPTER O. BACKGROUND AND INTRODUCTION

[Aug. 25: Simon Joyce]

This is a fast overview of graphs, signed graphs, and their equations and hyperplanes.

We begin with a few different definitions of a graph, for discursive purposes. All these definitions are popular, but in decreasing order. (Of course, the one we use is the least popular—and the most complicated. We can't help it.)

**Definition .1.** [Simple Graph][[LABEL D:0825simplegraph]] A *graph* is a pair  $\Gamma = (V, E)$ , where  $V$  is a set and  $E$  is a subset of  $\mathcal{P}_2(V)$ , the class of unordered pairs of (distinct) elements of  $V$ .

This definition doesn't account for things like loops, whose endpoints coincide, or parallel edges, which are edges with the same endpoints as each other, so we need to extend it for our purposes.

**Definition .2.** [Multigraph][[LABEL D:0825multigraph]] A *graph* is a pair  $\Gamma = (V, E)$ , where  $V$  is a set and  $E$  is a multisubset of  $\mathcal{P}_2(V)$ .

However, this definition still doesn't account for loops.

The following definition for a graph is what we will use.

**Definition .3.** [Graph][[LABEL D:0825graph]] A *graph* is a triple  $\Gamma = (V, E, I)$ , where  $V$  and  $E$  are sets and  $I$  is an incidence multirelation between  $V$  and  $E$  in which each edge has incidence of total multiplicity 2.

[Insert picture(s) of graphs here for instructional purposes.]

**Definition .4.** [Signed Graph][[LABEL D:0825signedgraph]] A *signed graph* is a graph whose edges have signs,  $+$  or  $-$ . Formally,  $\Sigma = (\Gamma, \sigma) = (V, E, I, \sigma)$ , where  $\sigma : E \rightarrow \{+, -\}$ .

[Insert picture(s) of signed graphs here.]

Starting at a vertex on a graph we can move along one of its incident edges to another vertex and repeat the process from the new vertex any number of times, to move around the graph in any way we please. To describe different ways of moving around a graph we use the following terms:

- A *path* has no repeated edges or vertices.
- A *trail* has no repeated edges.
- A *walk* may have repeated edges and/or vertices.
- A *circle* is a closed path, that is it has no repeated vertices or edges except the initial and the final vertex are the same.

Each edge of a graph implies an equation. The variables correspond to the vertices and an edge with endpoints  $v_i, v_j$  corresponds to the equation  $x_i = x_j$  in  $\mathbb{R}^n$ . The family of all hyperplanes corresponding to all edges,  $\mathcal{H}[\Gamma]$ , called the hyperplane arrangement generated

by  $\Gamma$ , divides up  $\mathbb{R}^n$  into regions that have a remarkable combinatorial meaning. For a signed graph, a positive edge  $+v_i v_j$  has hyperplane  $x_i = x_j$  and a negative edge  $-v_i v_j$  has hyperplane  $x_i = -x_j$ . We'll study the geometry of these arrangements of hyperplanes, both to learn more about the graph and to use the graph in order to understand the hyperplane arrangement.

All kinds of basic background information will be added during the lectures, both in the beginning of Chapter I and when needed as the lectures progress.

# CHAPTER I. GRAPHS

[27 August 2008: Nate Reff]

In this chapter we meet graphs, to develop the understanding and the technical background for signed graphs. Most of what we say about graphs will generalize later, to the more advanced topics of signed graphs, gain graphs, and even biased graphs.

## A. TECHNICAL DEFINITIONS

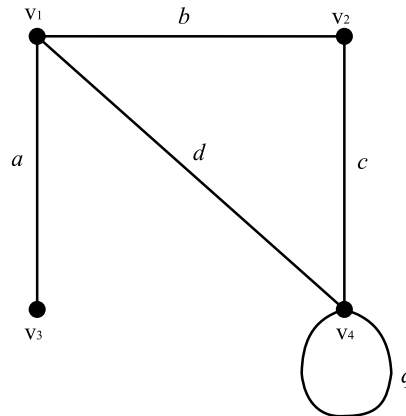
Here we meet the basic concepts and vocabulary of our version of graph theory.

### A.1. Definition of a graph.

We give a formal definition in terms of incidence between vertices and edges. This is rather heavy on notation, so we'll tend to ignore the technical statement in practice, but it's what we mean even when we don't mention it.

- An *incidence multi-relation*  $I$  between sets  $V$  and  $E$  is a multi-subset of  $V \times E$ .
- A *graph*  $\Gamma = (V, E)$  is an ordered pair consisting of sets  $V$  and  $E$  with an incidence multi-relation  $I$  between them such that every edge is incident to at most 2 vertices (not necessarily distinct).
- The elements of  $V$  are called the *vertices* of the graph  $\Gamma$ .
- The elements of  $E$  are called the *edges* of the graph  $\Gamma$ .

An example:



Notice that in the figure we have edge  $q$  incident to vertex  $v_4$  twice, so  $2 \cdot (v_4, q) \in I$ . This is consistent with our definition since we do not need edges to be incident to distinct vertices.

### A.2. Types of edge.

In the most general definition, there are four kinds of edge in a graph.

- A *loop* is an edge with two equal endpoints. A notation we often use is  $e:vv$ . Another is  $e_{vv}$ .

- A *link* is an edge with two distinct endpoints. A notation is  $e:vw$ . Another is  $e_{vw}$ .
- A *half edge* is an edge with one endpoint. A notation is  $e:v$ .
- A *loose edge* (or *free loop*) is an edge with zero endpoints. A notation is  $e:\emptyset$ .
- An *ordinary edge* is a link or a loop.

### A.3. Valuable notation.

- Always,  $n := |V|$ .
- Sometimes,  $m := |E|$ .
- $V(e)$  is the multiset of vertices of the edge  $e$ .
- Suppose  $S \subseteq E$ ; then  $V(S)$  is the set of endpoints of edges in  $S$ .
- $E^* := \{ \text{links and loops} \}$ , the set of ordinary edges.

### A.4. Types of graph.

There are two essential special kinds of graph:

- A *simple graph* is a graph in which all edges are links and there are no parallel edges (edges with the same endpoints).
- An *ordinary graph* is a graph with no half edges or loose edges; that is, all edges are ordinary.

Most graph theorists would call these the only kinds of graph. We will need half and loose edges later, when we generalize to signed graphs and even further; but in this chapter, graphs will be ordinary graphs unless we indicate otherwise.

### A.5. Types of subgraph.

- A *subgraph* of  $\Gamma$  is  $\Gamma'$  such that  $V' \subseteq V$ ,  $E' \subseteq E$ , has the same incidence multi-relation between  $V$  and  $E$ , every endpoint of every edge in  $E'$  is in  $V'$ , and each edge retains its type.
- Notice that  $\Gamma \setminus e = (V, E \setminus e)$ .
- The *deletion of a vertex set*, denoted by  $\Gamma \setminus X$  where  $X \subseteq V$ , is the subgraph with  $V(\Gamma \setminus X) := V \setminus X$ , and  $E(\Gamma \setminus X) := \{e \in E \mid \text{all endpoints of } e \text{ are in } V \setminus X\}$ .
- An *induced subgraph* of  $\Gamma$  is a subgraph of the following special form: Let  $X \subseteq V$ . The subgraph induced by  $X$  is  $\Gamma : X := (X, E : X)$ , where  $E : X := \{e \in E \mid \emptyset \neq V(e) \subseteq X\}$ . Notice that an induced subgraph has no loose edges.
- A *spanning subgraph* is a subgraph  $\Gamma'$  such that  $V' = V$ . ( $\Gamma'$  need not have any edges; it just must have all the vertices.)

### A.6. Contraction by an Edge.

The following cases cover the basics of *contracting an edge*. The graph  $\Gamma$  with an edge  $e$  contracted is denoted by  $\Gamma/e$ .

- *Case 1*: For a link  $e$  with vertices  $v$  and  $w$ ,  $\Gamma/e$  has  $v$  and  $w$  identified to a single vertex and  $e$  deleted. Sometimes the identified vertex will be denoted by  $v_e$ .
- *Case 2*: For a loop or loose edge,  $\Gamma/e = \Gamma \setminus e$ .
- *Case 3*: For a half edge  $e$  incident to vertex  $v$ , to get  $\Gamma/e$  we remove  $v$  and  $e$  but keep all other edges. A link  $f:vw$  becomes a half edge  $f:w$ . A loop  $f:vv$  or a half edge  $f:v$  becomes a loose edge  $f:\emptyset$ . All other edges remain as they were in  $\Gamma$ .



## B. BASIC STRUCTURES

### B.1. Connection.

- A *generalized path* is a sequence  $x_0x_1 \dots x_k$  where the  $x_i$ 's are alternately vertices and edges. If  $x_i$  is a link or a loop with endpoints  $v$  and  $w$  then  $\{x_{i-1}, x_{i+1}\} = \{v, w\}$  (note that these are multisets). If  $x_i$  is a half edge  $e:v$ , it is  $x_0$  or  $x_k$  so that  $x_0x_1 = ev$  or  $x_{k-1}x_k = ve$ . Lastly, if  $x_i$  is a loose edge the path is simply  $x_i$ .

Note that a “generalized path” is not necessarily a path (see below). I introduced it here only to explain how elements of a graph that are not necessarily vertices can be considered connected to each other.

- Two elements of  $\Gamma$ ,  $x$  and  $y$  (each of which may be a vertex or edge), are *connected* if there exists a generalized path containing both.

A standard theorem:

**Theorem B.1.** [[LABEL T:0827connvert]] *The relation of being the same or connected is an equivalence relation on  $V \cup E$ .*

[NEED DEFINITION OF "component". See next day?]

A *vertex component* is a component that has a vertex. Usually we just call these the *components* of the graph, because in this course we don't want to include loose edges in the count of components. If we ever do want to include loose edges, we need to say something special.

- We write  $c(\Gamma) :=$  the number of components, i.e., vertex components.

### [Aug 29: Jackie Kaminski]

We start with some definitions. Recall that  $V(e)$  is the set of endpoints of the edge  $e$ .

- *Walk*: A sequence  $v_0e_1v_1 \dots e_lv_l$  where  $V(e_i) = \{v_{i-1}, v_i\}$  and  $l \geq 0$ . (A walk of length zero is therefore just a vertex.)  
A *closed walk* is a walk where  $l \geq 1$  and  $v_0 = v_l$ .
- A *trail* is a walk with no repeated edges.
- A *closed walk* is a walk where  $l \geq 1$  and  $v_0 = v_l$ .
- A *path* is a trail with no repeated vertex. Sometimes it is called an *open path* to distinguish it from a closed path.
- A *closed path* is a closed trail with no repeated vertex other than that the last vertex is the first one. Despite the name, a closed path is not a path.
- The *length* of a walk, trail, or path is the number of edges in it, counted as many times as they appear in it.
- Two vertices are *connected* if there exists a path between them. (This is a special case of the definition of connection of vertices and edges in the last lecture.)

**Proposition B.2.** [[LABEL P:0829connequiv]] *The relation of being connected is an equivalence relation on  $V$ .*

The proof is basic graph theory and is left to the reader. It makes use of the next proposition.

**Proposition B.3.** [[LABEL P:0829walkconn]] *Vertices  $v, w$  are connected by a walk  $\iff$  they are connected by a path.*

The proof is basic graph theory and is left to the reader.

*Connected graphs.*

- A *connected component* (or *vertex component*, or simply *component*) of  $\Gamma$  is the subgraph induced by an equivalence class of the connectedness relation on  $V$ .  
An alternate definition of a (vertex) component of  $\Gamma$  is as a maximal connected subgraph that is not a loose edge.  
Thus, a loose edge is not a component. This is admittedly strange. Sometimes we might want a loose edge to be a component, so just in case, we define a *topological component* to be a vertex component or a loose edge.
- We say that  $\Gamma$  is *connected* if the relation of connection on  $V \cup E$  has exactly one equivalence class. Equivalently,  $\Gamma$  is *connected* if it is a loose edge, or it has no loose edges and the connection relation on  $V$  has exactly one equivalence class. That is,  $\Gamma$  is connected if it has exactly one component and no loose edges, or if it is a loose edge.
- The empty graph,  $\emptyset := (\emptyset, \emptyset)$  (that is, it has no vertices and no edges), is not connected.

## B.2. Degree.

**Definition B.1.** [[LABEL D:0829degree]] The *degree* or *valency* of a vertex, denoted by  $d(v)$ , is the number of edge ends incident with  $v$ .

To avoid getting lost in notation, we are not defining edge ends. Instead, we refer to the reader's intuition. Please note that a loop adds 2 to the valency and a link or half edge adds one to the valency of each endpoint.

See Figure A. [ADD FIGURES]

Notice that the common definition of the valency of  $v$  as the number of neighbors of  $v$  is only adequate for simple graphs.

**Definition B.2.** [[LABEL D:0829regular]] A *k-regular* graph is a graph where every vertex has degree  $k$ .

## B.3. Circles.

- A *circle* of  $\Gamma$  is a connected 2-regular subgraph of  $\Gamma$  which has at least one vertex, or its edge set. Another definition (equivalent to the first) is that a circle is the graph, or edge set, of a closed path.

For example, any loop is a circle, as is Figure B.

We require the subgraph to have a vertex in order to exclude loose edges as circles.

Please note that a closed path and the graph of a closed path are not the same thing. A closed path has a direction as well as a beginning point. The graph of a closed path has neither.

There is some ambiguity. Sometimes by a 'circle' we mean the edge set, sometimes the graph. The context should make the meaning clear.

#### B.4. Trees and their relatives.

Some basic definitions:

- A *tree* is a connected graph which does not contain a circle (as a subgraph).
- A *forest* is a graph which does not contain a circle (as a subgraph).  
Or equivalently, we can define a forest as a graph whose components are all trees.  
Please refer to Figure C.
- Observations:  
A tree is a connected forest.  
An empty graph (no vertices or edges) is a forest, but is *not* a tree. Recall that a connected graph must have exactly one connected component.
- A *spanning forest* is a spanning subgraph of  $\Gamma$  which is a forest.  
Observe that for any  $\Gamma = (V, E)$ , the graph  $(V, \emptyset)$  is a spanning forest for  $\Gamma$ .
- Similarly, a *spanning tree* a spanning subgraph of  $\Gamma$  which is a tree.  
Disconnected graphs do not contain any spanning trees.
- A *maximal forest* is a forest which is not properly contained in any other forest.  
Please refer to Figure C.

As an aside, please don't confuse *maximal*, which means not properly contained in any other object (or set) of the same type, with *maximum*, which means having the most elements. For forests in a graph, however, they come to the same thing.

**Theorem B.4.** [[LABEL T:0829maxforest]] *All maximal forests in  $\Gamma$  have the same number of edges, namely  $n - c(\Gamma)$ , where  $n = |V|$ .*

This theorem is elementary, yet not so easy to prove. (If you know matroid theory, notice that it is equivalent to the fact that every basis of the graphic matroid has the same size.)

**Theorem B.5.** [[LABEL T:0829forest]]

- (1) *A graph contains a spanning tree  $\iff$  the graph is connected.*
- (2) *A maximal forest consists of a spanning tree of each component of  $\Gamma$ .*

The proof is left to the reader.

Other tree-like graphs:

- A *1-tree* is a tree with one extra edge (not a loose edge). (We allow half edges in this definition.) See Figure D.
- *1-forest* is a graph where every component is a 1-tree.
- A *pseudo tree* is a graph which is a tree or a 1-tree.
- A *pseudo forest* is a graph in which every component is a pseudo tree.

#### B.5. Other special graphs.

- A *complete graph*, written  $K_n$ , is a simple graph in which every pair of vertices is adjacent.
- A *bipartite graph* is a graph whose vertex set has a bipartition  $V = V_1 \cup V_2$  such that every edge has one endpoint in  $V_1$  and the other in  $V_2$ . It need not be simple.
- A *complete  $k$ -partite graph* has vertices partitioned into  $k$  (non-empty) parts, and for vertices  $v, w$ , if  $v, w$  are in the same part, then are no  $vw$ -edges. And if  $v, w$  are in different parts, there is a  $vw$  edge. A complete  $k$ -partite graph with part sizes  $n_1, n_2, \dots, n_k$  is denoted by  $K_{n_1, n_2, \dots, n_k}$ .

Figure C shows a complete tripartite graph with tripartition  $\{x_1\}, \{v_1, v_2\}, \{w_1, w_2\}$ .

## B.6. Some special vertex sets.

- A *stable* or *independent* set of vertices is  $W \subseteq V$  such that  $E:W = \emptyset$ .  
In figure C,  $\{x_1\}$ ,  $\{v_1, v_2\}$ ,  $\{w_1, w_2\}$  are five stable sets.
- A *clique* is a vertex set whose members are pairwise adjacent.

## C. CONNECTEDNESS

(defined via paths). Components.  $c(?)$  = number of (vertex) components. (?).

## D. DELETION, CONTRACTION, AND MINORS

### D.1. Deletion.

We have already mentioned the two kinds of deletion. Deleting an edge set  $S$  simply means changing  $\Gamma = (V, E)$  to  $\Gamma \setminus S := (V, E \setminus S)$ . Deleting a vertex set  $X$  means deleting not only the elements of  $X$  but also the edges that have an endpoint in  $X$ ; this is written  $\Gamma \setminus X$ . Both kinds of deletion are ways of taking a subgraph.

[Aug 29: Jackie Kaminski]

### D.2. Contraction.

(Most notes are from an earlier class.)

- Notice that we already defined how to contract a link, loop, half edge, and loose edge.  
We are now restricting ourselves to ordinary graphs again.
- Refer to Figure D for a visual representation of contraction by a single edge.
- Contraction by an edges set  $S \subseteq E$ , is denoted  $\Gamma/S = (V/S, E \setminus S)$ , and is equivalent to a sequence of edge contractions by the edges in  $S$ . It can be shown that the resulting graph is the same regardless of the order in which the edges are contracted (provided you aren't too pedantic about the naming of vertices in the resulting graph). Proving this certainly takes some work but is left to the reader.
- See Figure E for an example
- For a graph  $\Gamma$ , let  $\pi(S) =$  the partition of  $V$  s.t. each part is the vertex set of a (connected) component of  $(V, S)$ . In other words  $V/S$  is  $\pi(S)$ . Furthermore we will let  $[v]$  denote the part of  $\pi(S)$  containing the vertex  $v$ .
- See Figure F
- An edge  $f$  of the contraction  $\Gamma/S$  is  $f \in E \setminus S$ , and for  $V(f) = \{v, w\}$ ,  $f$  in  $\Gamma/S$  has endpoints  $[v], [w]$ .

[3 Sept. 2008: Yash Lodha]

### D.3. Minors.

A *minor* of  $\Gamma$  is defined as a contraction of a subgraph of  $\Gamma$ . It turns out that the order of contracting and taking subgraphs makes no difference.

**Theorem D.1.** *Any graph obtained from a graph  $\Gamma$  by a series of edge contractions and deletions and vertex deletions is a minor of  $\Gamma$ .*

We'll prove more general theorems later, in Chapters II and III, so I omit the proof here.

The following theorem is one of the main ways in which minors are used. It characterizes the graphs that embed in a surface in terms of forbidden minors. Each successive part is much harder to prove. The general name for these results is “Kuratowski-type theorems”.

**Theorem D.2.** [[LABEL T:0903kuratowski]] *Let  $\Gamma$  be a graph.*

- (1) [Kuratowski (mainly) and Wagner]  $\Gamma$  is planar iff  $\Gamma$  does not contain  $K_5, K_{3,3}$  as minors.
- (2) [Archdeacon, Glover, and Huneke]  $\Gamma$  is projective planar iff  $\Gamma$  does not contain a list of 35 graphs as minors.
- (3) [Robertson and Seymour]  $\Gamma$  embeds in a surface  $S$  iff  $\Gamma$  does not contain any of a finite list of graphs, which depends on  $S$ , as a minor.

## E. CLOSURE AND CONNECTED PARTITIONS

We now remind ourselves of the definition of an abstract closure operator on  $E$ .

**Definition E.1.** [[LABEL D:0903absclosure]] A *closure operator* on  $E$  is a function  $\mathcal{P}(E) \rightarrow \mathcal{P}(E) : S \mapsto \overline{S}$  such that the following axioms hold for subsets  $S$  and  $T$  of  $E$ :

- (1)  $S \subseteq \overline{S}$ .
- (2)  $S \subseteq T \implies \overline{S} \subseteq \overline{T}$ .
- (3)  $\overline{\overline{S}} = \overline{S}$ .

A set  $S \subseteq E$  is a *closed set* if  $S = \overline{S}$ .

The closed sets when ordered by inclusion form a partially ordered set (poset).

The closure operator in a graph also obeys a very important fourth property, the *exchange property*:

- (4) Let  $S$  be a closed subset of  $E$ . If  $e, f \notin S$  and  $e \in \overline{f \cup S}$ , then  $f \in \overline{e \cup S}$ .

We define

$$\Pi_V := \{ \text{all partitions of } V \} \quad \text{and} \quad \Pi_n := \Pi_{[n]}.$$

Partitions of a set are partially ordered by refinement; that is,  $\pi \leq \tau$  if each part of  $\pi$  is contained in a part of  $\tau$ . (Recall that a partition may not have an empty part.) We say  $\pi \in \Pi_V$  is *connected* (in  $\Gamma$ ) if each part  $B \in \pi$  induces a connected subgraph. Let

$$\Pi(\Gamma) := \text{the set of connected partitions of } V.$$

We define the *partition of  $V$  induced by an edge set  $S$*  as  $\pi(S) := \pi(V, S) :=$  the partition of  $V$  into the subsets which are the vertex sets of the connected components of  $S$ —i.e., of  $(V, S)$ . That is, the parts are the equivalence classes of the connection relation of  $(V, S)$ . Closed sets are intimately related to connected partitions.

**Lemma E.1.** [[LABEL L:0903closureptn]]  *$S$  is closed iff it equals  $\bigcup_{B \in \pi} E: B$  for some  $\pi \in \Pi(\Gamma)$ .*

*Proof.* [THERE SHOULD BE A PROOF.] □

**Theorem E.2.** *The poset of closed sets ordered by inclusion is isomorphic to the poset  $\Pi(\Gamma)$  of connected partitions of  $\Gamma$  ordered by refinement.*

*Proof.* [THERE SHOULD BE A PROOF.] □

**E.1. Equivalent perspectives on the closure of an edge set.**

Recall from Definition E.1 that an abstract closure operator on a set  $X$  is a function from the  $\mathcal{P}(X)$  into  $\mathcal{P}(X)$  that is extensive, increasing, and idempotent. We now define a function from  $\mathcal{P}(E) \rightarrow \mathcal{P}(E)$  that we will name  $\text{clos}$ . We leave it up to the reader to verify that this function is in fact an abstract closure operator.

**Definition E.2.** [[LABEL D:0908closure]] In an ordinary graph  $\Gamma$ , for  $S \subseteq E$ , the *closure* of  $S$  is  $\text{clos}(S) := S \cup \{e : \text{the endpoints of } e \text{ are connected in } S\}$ .

Notice that it is redundant to list  $S$  in the definition of  $\text{clos}(S)$ , since the endpoints of an edge of  $S$  are always connected in  $S$ .

**Definition E.3.** [[LABEL D:0908closed]] We say  $S \subseteq E$  is *closed* if  $\text{clos}(S) = S$ .

Recall that  $S:B_i$  is the set of edges in  $S$  with all of their endpoints in the vertex set  $B_i$ . (We may sometimes abuse notation and write  $S:B_i$  as shorthand for  $(B_i, S:B_i)$ .) Along these lines we write  $c(S:B_i)$  as shorthand for  $c(B_i, S:B_i)$ , the number of connected (vertex) components in the subgraph induced by vertex set  $B_i$ . In other words, induced subgraphs only contain the inducing vertices, not all the vertices of  $\Gamma$ .

**Proposition E.3.** [[LABEL P:0908indclosure]] Let  $\pi(S) = \{B_1, \dots, B_k\}$ . Then

$$\text{clos}_\Gamma(S) = \bigcup_{i=1}^k \text{clos}_{\Gamma:B_i}(S:B_i) = \bigcup_{i=1}^k E:B_i.$$

*Proof Sketch.* It's immediate that  $\text{clos}_{\Gamma:B_i}(S:B_i) = E:B_i$ , and therefore  $\bigcup_{i=1}^k \text{clos}_{\Gamma:B_i}(S:B_i) = \bigcup_{i=1}^k E:B_i$ . Notice that  $\text{clos}_\Gamma(S) \supseteq \bigcup_{i=1}^k E:B_i$ , since any edge  $e \in E:B_i$  has its endpoints connected in  $S$  by the definition of  $\pi(S)$ . Furthermore,  $\text{clos}_\Gamma(S) \subseteq \bigcup_{i=1}^k E:B_i$  since any edge  $e$  with one endpoint in  $B_i$  and the other in  $B_j$  ( $i \neq j$ ) can't possibly have its endpoints connected in  $S$ , since  $S:B_i$  and  $S:B_j$  are separate components of the subgraph  $(V, S)$ . (This last step should be fleshed out by the reader.)  $\square$

**Proposition E.4.** [[LABEL P:0908meetjoin]] For any abstract closure operator on  $E$ , the class of closed subsets forms a lattice with meet and join defined as follows: For  $S, T$  closed subsets of  $E$ ,  $S \wedge T = S \cap T$  and  $S \vee T = \text{clos}(S \cup T)$ .

So in particular, this holds for the closure operator we've been working with on the edge subsets of  $\Gamma$ .

**E.2. Edge sets induced by partitions.**

Recall that  $\Pi(\Gamma)$  is the set of all connected partitions of  $V$ , i.e. for  $\pi \in \Pi(\Gamma)$  and  $B \in \pi$ , any two vertices in  $B$  are connected in  $\Gamma:B$ . We notice immediately that for any  $S \subseteq E$ , the partition  $\pi(S) \in \Pi(\Gamma)$ . This observation allows us to define a function  $\pi : \mathcal{P}(E) \rightarrow \Pi(\Gamma)$  by  $S \mapsto \pi(S)$ . We now present a definition followed by a lemma about  $\pi$ .

**Definition E.4.** [[LABEL D:0908Epi]] For any partition  $\pi$  of  $V$ ,  $E(\pi) := \bigcup_{B \in \pi} E:B$ .

Notice that when  $\pi$  is not a connected partition many of the terms in the union will be empty. The following lemma is only for connected partitions.

**Lemma E.5.** [[LABEL L:0908clos]] For each  $\pi \in \Pi(\Gamma)$ ,  $\boldsymbol{\pi}(E(\pi)) = \pi$ . Furthermore,  $E(\pi(S)) = \text{clos}(S)$ .

Thus, from  $\pi(S)$  we can't in general recover  $S$ , but we can always recover  $\text{clos}(S)$ .

**Corollary E.6.** [[LABEL C:0908piofclos]]  $\pi(\text{clos}(S)) = \pi(S)$ .

*Proof.* Let  $\pi(S) = \{B_1, \dots, B_k\}$ . From Proposition E.3,  $\text{clos}(S) = \bigcup_{i=1}^k E:B_i$ . Each part in  $\pi(\text{clos}(S))$  will be the vertex set of a maximal connected component of  $\bigcup_{i=1}^k E:B_i$ . These are precisely the sets  $B_i$ .  $\square$

**Corollary E.7.** [[LABEL C:0908piEpi]] For any  $S \subseteq E$ ,  $\boldsymbol{\pi}(E(\pi(S))) = \pi(S)$ .

*Proof.*  $E(\pi(S)) = \bigcup_{B \in \pi(S)} E:B$  by definition, and  $\boldsymbol{\pi}(E(\pi(S)))$  is  $\pi(\bigcup_{B \in \pi(S)} E:B)$ , which is precisely  $\pi(S)$  since each  $E:B$  is connected.  $\square$

### E.3. Lattices.

Whenever  $S \subseteq S' \subseteq E$ , then  $\pi(S)$  is a refinement of  $\pi(S')$ , that is to say, each of the parts of  $\pi(S)$  is contained in a part of  $\pi(S')$ . Readers familiar with partitions of a set  $V$  will think of the last statement as  $\pi(S) \leq \pi(S')$ ; this defines a partial ordering of partitions called the *refinement ordering*. It is well known that the set  $\Pi(V)$  of all partitions of  $V$  with the refinement ordering forms a lattice. It is left to the reader to check that the set of connected partitions also forms a lattice in which the meet operation is the same as in  $\Pi(V)$  and the join operation is  $\tau \vee \tau' = \bigwedge \{\pi \in \Pi(\Gamma) : \pi \geq \tau, \tau'\}$ .

When  $\tau, \tau'$  are two partitions of  $V$  such that  $\tau \leq \tau'$  (ordered by refinement), then  $E(\tau) \subseteq E(\tau')$ . Here we remind the reader that  $\mathcal{P}(E)$ , ordered by set inclusion, is also a lattice with the intersection and union operations. These observations and the following definition leads us to our next theorem.

**Definition E.5.** [[LABEL D:0908lattice]]  $\text{Lat}(\Gamma)$  is the class whose members are the closed edge sets of  $\Gamma$ , ordered by containment.

**Theorem E.8.** [[LABEL T:0908LatIso]]  $\Pi(\Gamma) \cong \text{Lat}(\Gamma)$ . Specifically,  $\boldsymbol{\pi} : \text{Lat}(\Gamma) \rightarrow \Pi(\Gamma)$  is an order isomorphism.

*Proof.* We already noted that  $S \subseteq S' \subseteq E \implies \pi(S) \leq \pi(S')$  and that for  $\tau, \tau' \in \Pi(\Gamma)$ ,  $\tau \leq \tau' \implies E(\tau) \subseteq E(\tau')$ . So all that's left to show is that  $\boldsymbol{\pi}$  is a bijection between the connected (vertex) partitions of  $\Gamma$  and the closed (edge) subsets of  $\Gamma$ .

To see that  $\boldsymbol{\pi}$  is injective, let  $S, S'$  be closed subsets of  $E$ , and assume  $\pi(S) = \pi(S')$ . By Proposition E.3,  $S = \bigcup_{B \in \pi(S)} E:B = \bigcup_{B \in \pi(S')} E:B = S'$ . To see that  $\boldsymbol{\pi}$  is surjective, we notice that for  $\tau$  a connected partition of  $\Gamma$ ,  $E(\tau)$  is closed by Proposition E.3. This completes our proof.  $\square$

## F. INCIDENCE AND ADJACENCY MATRICES

[3 Sept. 2008: Yash Lodha]

We now define the adjacency matrix  $A(\Gamma)$ . This is an  $n \times n$  or  $V \times V$  matrix where  $V(\Gamma) = \{v_1, v_2, \dots, v_n\}$ . Each entry  $a_{i,j} = 1$  if  $v_i \sim v_j$  and  $a_{i,j} = 0$  if  $v_i \not\sim v_j$ . The incidence

matrix  $H(\Gamma)$  is a  $V \times E$  matrix. The entry  $\eta_{i,j} = 0$  if the edge  $e_j$  is not incident with the vertex  $v_i$  and  $\eta_{i,j} = 1$  if  $e_j$  is incident with  $v_i$ . The degree matrix  $D(\Gamma)$  is a  $V \times V$  diagonal matrix where the entry  $d_{i,i}$  is the degree of the vertex  $v_i$ .

**Theorem F.1.** [[LABEL T:0903incidence-adjacency]]  $HH^T = D - A$ .

*Proof.* We check the cases  $i \neq j$  and  $i = j$  separately when multiplying the  $i$ -th row of  $H$  with the  $j$ -th column of  $H^T$ . The rest follows easily.  $\square$

\*\*\* Eigenvalues.

## G. ORIENTATION

[September 5: Yash Lodha]

If we have an unoriented graph  $\Gamma$ , we give it an orientation by giving every edge a direction. We write  $\vec{\Gamma}$  for an orientation of  $\Gamma$ . The orientation is not a property inherent in the graph  $\Gamma$ , but a property superimposed on the graph for various purposes. In the case of directed graphs, however, the property of being ‘directed’ is inherent in the graph.

- *Cycle*: In an oriented graph a cycle is a directed closed path, or equivalently a circle that is oriented so each vertex is ‘coherent’ or ‘consistent’ (that is, the two edge directions agree).
- *Directing a circle*: Give the circle as a whole a direction. (This is a completely separate property of the circle from directions on the edges).

By ordering  $V$ , we get an acyclic orientation, if  $\Gamma$  has no loops. This orientation is unique.

**Theorem G.1.** [[LABEL T:0903acyclic]] *Every acyclic orientation arises in this way, i.e., from a linear ordering of  $V$ .*

Hence there is an equivalence in that statement above. It is important to note that some linear orderings may yield the same acyclic orientation.

In an oriented graph there can be two special kinds of vertices.

- *Sink*: A vertex with only entering edges.
- *Source*: A vertex with only departing edges.

These definitions motivate the following lemma.

**Lemma G.2.** [[LABEL L:0903sourcesink]] *Every acyclic orientation has a source and a sink.*

*Proof.* We start on an edge and walk along a path following edge directions. If we repeat a vertex we form a cycle, which contradicts the assumption that our graph is acyclic. If we never repeat a vertex in our path, then since  $|V|$  is finite we must end our path at a vertex that only has entering edges. This proves the existence of a sink.

To prove the existence of a source, reverse the orientations of all edges. A sink in the reversed graph is a source in the original orientation. Alternatively, apply the previous argument in reverse.  $\square$



*Proof of Theorem G.1.* We perform induction on  $|V|$ . If  $\vec{\Gamma}$  is acyclic, then it must have a sink  $s$ . Then by our inductive hypothesis  $\vec{\Gamma} \setminus s$  is acyclic and has an ordering  $v_1 < v_2 < \dots < v_{n-1}$ . The ordering for  $\vec{\Gamma}$  is  $v_1 < v_2 < \dots < v_{n-1} < s$ .  $\square$

Now we observe that a complete ordering is not necessary. [I will discuss an example once I learn graphics in Tex.] In fact a partial ordering can suffice for providing us with a corresponding acyclic orientation of the graph.

**Theorem G.3.** *For each  $\vec{\Gamma}$ , there exists a smallest partial ordering of  $V$  that gives the orientation  $\vec{\Gamma}$ . The linear orderings that give  $\vec{\Gamma}$  are precisely the linear extensions of that smallest partial ordering.*

An important example is the complete graph.

**Example G.1.** [[LABEL X:0903kn]] *Acyclic orientations of  $K_n$ .* Every partial ordering of  $V$  that gives  $K_n$  as its comparability graph is a chain (a total ordering). There are  $n!$  of these, one for each permutation of  $V$ .

**Corollary G.4.** *The acyclic orientations of  $K_n$  correspond bijectively to the permutations of  $V$  in a natural way.*

*Proof.* The correspondence is that a total ordering of  $V$  implies an orientation of each edge from lower to higher.

Conversely, suppose  $K_n$  is acyclically oriented. Then there is a corresponding partial ordering of  $V$ , but it's a total ordering because every pair of vertices is comparable.  $\square$

**Example G.2.** [[LABEL X:0903compar]] *Comparability graph:* This is the graph of all comparability relations in a poset. This means that the vertex set is the set of elements of the poset, and we connect elements  $u, v$  with an edge if  $u, v$  are comparable in our poset. There's an extensive literature on comparability graphs. A good, readable source is Golumbic's [PG].

An orientation that is not totally acyclic is called *cyclic*. But we can also have a *totally cyclic orientation*, where every edge is in a cycle.

**Proposition G.5.**  *$\Gamma$  has an acyclic orientation iff it has no loops.  $\Gamma$  has a totally cyclic orientation iff it has no isthmus (or bridge).*

An oriented graph has an incidence matrix.

**Definition G.1.** [[LABEL D:0903orincidence]] An *oriented incidence matrix* of a graph is the incidence matrix of any orientation of that graph. An *incidence matrix of an orientation* of a graph has, for each edge  $e$ , in the column denoted by  $e$ , an entry of  $+1$  at the row of its head vertex and an entry of  $-1$  at the tail.

There are many different oriented incidence matrices of a graph, in fact,  $2^{m'}$  where  $m'$  is the number of links (and half edges).

## H. EQUATIONS AND INEQUALITIES FROM EDGES

[September 8: Jackie Kaminski]

## H.1. Arrangements of hyperplanes.

Now we think of the edge set of  $\Gamma$  as  $\{v_1, \dots, v_n\}$ , and we begin by considering only ordinary graphs  $\Gamma$ . We define

$$h_{ij} := \{x \in \mathbb{R}^n \mid x_i = x_j\}.$$

When  $i \neq j$ ,  $h_{ij}$  is clearly a hyperplane (a codimension-1 linear subspace) of  $\mathbb{R}^n$ . We will refer to  $h_{ii}$ , which is all of  $\mathbb{R}^n$  since it corresponds to the equation  $x_i = x_i$ , as the “degenerate hyperplane”, because it will be convenient later to allow it as one of a family of hyperplanes.

**Definition H.1.** [[LABEL D0908hyp]] An *arrangement of hyperplanes* is a finite set (or multiset) of hyperplanes in  $\mathbb{R}^n$ .

**Definition H.2.** [[LABEL D0908HypGamma]] The hyperplane arrangement induced in  $\mathbb{R}^n$  by  $\Gamma$ ,  $\mathcal{H}[\Gamma]$ , is the multiset of hyperplanes  $\{h_{ij} \mid e:v_iv_j \in E\}$ . (Recall that  $n = |V|$ .)

We notice that each loop in  $\Gamma$  corresponds to the degenerate hyperplane. And furthermore we note the obvious correspondence between the multiset  $\mathcal{H}[\Gamma]$  and the edges of  $\Gamma$ . In fact there are many equivalent points of view we can take, as we notice the following (bijective) correspondences, that we describe on elements, but they extend naturally to their respective sets.

- The edge  $e:v_iv_j \leftrightarrow$  the equation  $x_i = x_j$ .
- $x_i = x_j \leftrightarrow$  the hyperplane  $h_{ij}$  in  $\mathbb{R}^n$ , by geometry.
- $e:v_iv_j \leftrightarrow$  column  $c_e$  in  $H(\Gamma)$ . Recall that  $H(\Gamma)$  is the incidence matrix of  $\Gamma$ . This correspondence is immediate from the definition of  $H$ .
- Column  $c_e$  in  $H(\Gamma) \leftrightarrow x_i = x_j$  by vector space duality.

Before looking into further correspondences, we set up a bit more terminology.

**Definition H.3.** [Region of  $\mathcal{A}$ ] [[LABEL D:0908region]] For an arrangement  $\mathcal{A}$  of hyperplanes in  $\mathbb{R}^n$ , a *region* of  $\mathcal{A}$  is a connected component of  $\mathbb{R}^n \setminus \bigcup_{A \in \mathcal{A}} A$ . Thus, if there is a degenerate hyperplane in  $\mathcal{A}$ , then  $\mathcal{A}$  has no regions.

Now we define

$$\mathcal{L}(\mathcal{A}) := \{\bigcap \mathcal{S} \mid \mathcal{S} \subseteq \mathcal{A}\},$$

which we will later see is a lattice, and we will later have a theorem saying  $\mathcal{L}(\mathcal{H}[\Gamma]) \cong \text{Lat}(\Gamma) \cong \Pi(\Gamma)$ , where the lattice isomorphisms are all natural. This will eventually allow us to switch between the perspectives of geometry, lattices, and graphs. Furthermore we can think of any of the correspondences above as correspondences between subsets instead of between individual elements.

Finally, we close with two lemmas that we will revisit later.

**Lemma H.1.** [[LABEL L:0908closspan]] For  $e \in \text{clos}(S)$ ,  $c_e \in \langle c_f : f \in S \rangle$ .

**Lemma H.2.** [[LABEL L:0908hypintersection]] For  $S \subseteq E$ ,  $\bigcap \mathcal{H}[S] = \bigcap \mathcal{H}[\text{clos}(S)]$ .

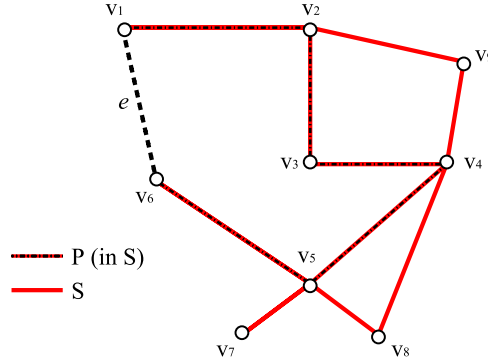
This second lemma is the vector dual of the first.

[September 10, 2008: Nate Reff]

## H.2. Graphic Hyperplane Arrangements and the Intersection Lattice.

**Lemma H.3.** [[LABEL L:0910lemma1a]]  $e \in \text{clos}(S) \implies c_e \in \langle c_f : f \in S \rangle$ .

*Proof.* Let's draw a nice picture to see how things work.



The red lines denote edges of  $S \subseteq E$  in a graph  $\Gamma = (V, E)$ . If  $e \in (\text{clos}(S) \setminus S)$  as in the picture, then there exists a path  $P \subseteq S$  such that there is a circle. We will show that  $\langle c_f : f \in S \rangle$ . Because  $e \in (\text{clos}(S) \setminus S)$  and thus  $e \in \text{clos}(S)$ , there exists a path  $P = v_1 v_2 \cdots v_l$  connecting the two endpoints of  $e$ . Now let's label the vertex set in such a way that we start at  $v_1$ , one endpoint of  $e$  and traverse  $P$  until we reach the other endpoint of  $e$ ,  $v_l$  (in our particular example,  $v_6$ ). Then arbitrarily assign the remaining vertices. If we do this then the columns of  $P \cup e$  are the following:

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix},$$

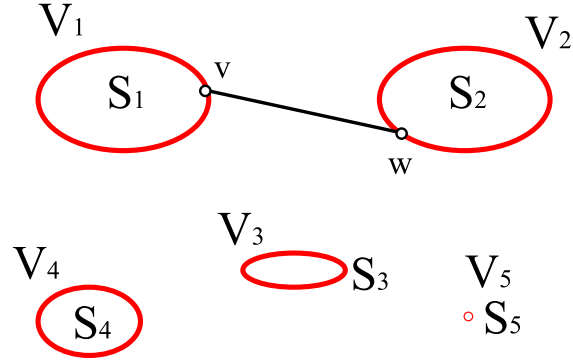
where the columns of the matrix correspond to  $\{e_1, e_2, \dots, e_l, e\}$  and the rows correspond to  $v_1, v_2, \dots, v_l, v_{l+1}, \dots$

Then  $c_e = c_{e_2} + c_{e_3} + \cdots + c_{e_l}$ , so  $c_e$  is spanned by the column vectors of edges in  $S$ .  $\square$

**Lemma H.4.** [[LABEL L:0910lemma1b]]  $c_e \in \langle c_f : f \in S \rangle \implies e \in \text{clos}(S)$ .

*Proof.* Suppose  $e \notin \text{clos}(S)$ . Then the endpoints of  $e$  belong to different components of  $(V, S)$ , simply because there is no path in  $S$  connecting the endpoints.

Now, for a working example, let's consider the following graph  $\Gamma$ :



The incidence matrix  $H(\Gamma)$  looks like this:

$$\begin{bmatrix}
 & (S_1) & (S_2) & (S_3) & (S_4) & (S_5) & \begin{pmatrix} (e) \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} & (S^c \setminus e) \\
 (V_1) & H(S_1:V_1) & O & O & O & O & & * \\
 (V_2) & O & H(S_2:V_2) & O & O & O & \begin{pmatrix} 0 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{pmatrix} & * \\
 (V_3) & O & O & H(S_3:V_3) & O & O & \mathbf{0} & * \\
 (V_4) & O & O & O & H(S_4:V_4) & O & \mathbf{0} & * \\
 (V_5) & O & O & O & O & H(S_5:V_5) & \mathbf{0} & *
 \end{bmatrix},$$

where the columns of the matrix are indexed by the edges of  $S_1, S_2, S_3, S_4, S_5, e$ , and  $S^c \setminus e$ ; the rows of the matrix are indexed by the sets  $V_1, V_2, V_3, V_4, V_5$ ; and the column of  $*$ 's stands for  $H(S^c \setminus e)$ . The nonzero entries in column  $c_e$  are, in the rows of  $V_1$ , in row  $v$ , and in the rows of  $V_2$ , in row  $w$ .  $O$  is a zero matrix, and  $\mathbf{0}$  is a column vector of zeros.

Now we return to the general proof. Suppose  $e:vw$  has  $v \in V_1$  and  $w \in V_2$ , and that there is a sum  $\sum_{e_i \in S} \alpha_i c_{e_i} = c_e$ . The edges in a component  $S_j$  of  $S$  which doesn't contain an endpoint of  $e$  have to add up to zero in the sum, so they can be ignored. Thus, looking only at the rows of  $V_1$ ,

$$\sum_{e_i \in S_1} \alpha_i c_i + \sum_{e_i \in S_2} \alpha_i c_i = c_e,$$

where for brevity we write  $c_i$  for the column of  $e_i$ .

Looking only at the rows of  $V_1$ , we note two facts. First, let  $c'_i$  and  $c'_e$  denote just the  $V_1$  rows of  $c_i$  and  $c_e$ . Then

$$(H.1) \quad \text{[[LABEL E:0910S1]]} \sum_{e_i \in S_1} \alpha_i c'_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Second, all columns in  $S_1$ , restricted to the rows of  $V_1$ , have entries that sum to zero, so if we add up all the rows in Equation (H.1), the left-hand side of the equation sums to 0 and the right-hand side sums to 1. This is a contradiction! Hence there does not exist a linear combination which is equal to  $e$ . Therefore we can say that  $e \in \text{clos}(S)$ .  $\square$

**Lemma H.5.** [[LABEL L:0910lemmaSubLemma]] *For a hyperplane  $H_e \in \mathcal{H}[\Gamma]$ ,  $\bigcap \mathcal{H}[S] \subseteq H_e \iff e \in \text{clos}(S)$ .*

**Lemma H.6.** [[LABEL L:0910lemma2]]  $\bigcap \mathcal{H}[S] = \bigcap \mathcal{H}[\text{clos}(S)]$ .

*Proof.* Use Lemma H.5, and dualize Lemmas H.3 and H.4.  $\square$

We define a subset  $S \subseteq E$  to be *dependent* if there exists an  $e \in S$  such that  $e \in \text{clos}(S \setminus e)$ .

**Proposition H.7.** [[LABEL P:0910prop1]]  *$S$  is independent  $\iff S$  is a forest.*

*Proof.* This is immediate from the definition of closure.  $\square$

**Theorem H.8.** [[LABEL T:0910thm1]] *Let  $S \subseteq E$ .  $S$  is independent in  $\Gamma$  (so  $S$  is a forest)  $\iff$  the columns of  $S$  in  $H(\Gamma)$  are linearly independent.*

*Proof.* Immediate corollary of Lemmas H.3 and H.4.  $\square$

We define a *linearly closed set of columns* to be the intersection of  $\{c_e : e \in E\}$  with a subspace of  $F^n$ .

**Corollary H.9.** [[LABEL C:0910cor1]] *The closed edge sets  $\leftrightarrow$  the linearly closed sets of columns of  $H(\Gamma)$ .*

**Theorem H.10.** [[LABEL T:0910thm2]]  $\Pi(\Gamma) \cong \text{Lat}(\Gamma) \cong \{\text{linearly closed sets of columns}\} \cong \mathcal{L}(\mathcal{H}[\Gamma])$ .

*Proof.* This follows from the relationships we've already seen among the various lattices and closures.  $\square$

### H.3. Regions and Orientations.

An orientation of  $\Gamma$  defines a positive side of each hyperplane  $h_{ij} \in \mathcal{H}[\Gamma]$ , called the *positive half-space* of the hyperplane. If we orient  $e: v_i v_j$  from  $v_i$  to  $v_j$ , the positive half-space is the set  $\{x \in \mathbb{R}^n : x_i < x_j\}$ . For each orientation, therefore, there is a family of positive half-spaces.

**Lemma H.11.** [[LABEL L:0910lemma3]] *A cyclic orientation of  $\Gamma$  gives an empty intersection of positive half-spaces.*

*Proof.* [MUST PROVE IT]  $\square$

Thus, any region is the intersection of positive half-spaces in a unique orientation of  $\Gamma$ , which is necessarily acyclic.

**Theorem H.12.** [[LABEL T:0910thm3]] *The intersection of positive half-spaces of an orientation of  $\Gamma$  is empty if the orientation is cyclic, but it is a region of  $\mathcal{H}[\Gamma]$  if the orientation is acyclic.*

*Proof.* In the cyclic case we just use Lemma H.11. In the acyclic case the orientation corresponds to a linear ordering of vertices, say  $v_1 < v_2 < \dots < v_n$ . Then  $(1, 2, \dots, n)$  will be in the intersection of positive half-spaces. Therefore the intersection is nonempty, and in fact a region.  $\square$

o Graphic hyperplane arrangements, intersection lattice. o Linear dependence of columns of the incidence matrix. o Closure, intersection lattice, and connected partitions. o Regions; consistency of inequalities via acyclic orientation.

## I. COLORING

[September 12: Simon Joyce]

Given a graph  $\Gamma$ , a *coloration* (or *coloring*) of  $\Gamma$  in  $k$  colors is a function  $\gamma : V \rightarrow \Lambda$ , a set of  $k$  colors. It doesn't matter for the definition exactly which  $k$ -element set  $\Lambda$  is, but often enough it is best to choose it to be the set  $[k] := \{1, 2, \dots, k\}$  of the first few positive integers.

An edge  $e:vw$  is *proper* if  $\gamma(v) \neq \gamma(w)$  and a coloration is *proper* if every edge is proper. For example, a graph with a loop can't ever be properly colored. Any coloration  $\gamma$  of a graph  $\Gamma$  has a set of proper edges and a set of improper edges. We will call the set of improper edges  $I(\gamma)$ . We say a graph is *k-colorable* if there exists a proper coloration in  $k$  colors.

**Definition I.1.** [[LABEL D:0912 chrom num]] For a graph  $\Gamma$  we define its *chromatic number* to be

$$\chi(\Gamma) = \min\{k : \Gamma \text{ is } k\text{-colorable}\}.$$

For instance,  $\chi(K_n) = n$  and  $\chi(\bar{K}_n) = 1$  for  $n \geq 1$ . For a forest  $F$  with at least one edge,  $\chi(F) = 2$ . In fact, for any bipartite graph that has at least one edge,  $\chi(\Gamma) = 2$ . At the opposite extreme,  $\chi(\Gamma) = \infty$  if, and only if,  $\Gamma$  has a loop.

## J. CHROMATIC FUNCTIONS

[September 12: Simon Joyce]

### J.1. The Chromatic Polynomial.

We now turn our interest to the number of proper colorations of a graph  $\Gamma$  in  $\lambda$  colors. We define the quantity

$$\chi_\Gamma(\lambda) := \text{the number of proper colorings of } \Gamma \text{ in } \lambda \text{ colors,}$$

where  $\lambda$  is a positive integer. In order to prove results about  $\chi_\Gamma(\lambda)$  let's define the set  $P_\Gamma = \{\text{proper colorings of } \Gamma\}$ .

**Lemma J.1.** [[LABEL L:0912 chrom dc]]

$$\chi_\Gamma(\lambda) = \chi_{\Gamma \setminus e}(\lambda) - \chi_{\Gamma/e}(\lambda)$$

for  $\lambda \in \mathbb{Z}_{>0}$ .

*Proof.* If  $e$  is a loop the result is clear because the left-hand side equals 0 and on the right-hand side  $\Gamma \setminus e = \Gamma/e$ . If  $e$  is a link, first observe that  $P_\Gamma \subseteq P_{\Gamma \setminus e}$ . Consider the set  $P_{\Gamma \setminus e} \setminus P_\Gamma$ :

$$\begin{aligned} P_{\Gamma \setminus e} \setminus P_\Gamma &= \{\text{proper colorings of } \Gamma \setminus e \text{ which are improper for } \Gamma\} \\ &= \{\text{proper colorations of } \Gamma \setminus e \text{ in which the endpoints of } e \text{ have the same color}\}. \end{aligned}$$

So there is a natural bijection from the set  $P_{\Gamma \setminus e} \setminus P_\Gamma$  to the set  $P_{\Gamma/e}$ , under which  $v_e \in \Gamma/e$  gets the same color as that of both endpoints of  $e \in \Gamma \setminus e$ . We conclude that  $|P_{\Gamma \setminus e}| = |P_\Gamma| + |P_{\Gamma/e}|$  and the result follows.  $\square$

**Lemma J.2.** [[LABEL L:0912 chrom mult]]

$$\chi_{\Gamma_1 \cup \Gamma_2}(\lambda) = \chi_{\Gamma_1}(\lambda)\chi_{\Gamma_2}(\lambda) \text{ where } \lambda \in \mathbb{N}.$$

*Proof.* Obvious. [IT WOULDN'T HURT TO GIVE A PROOF!]  $\square$

**Theorem J.3.** Given a graph  $\Gamma$  with no loops then  $\chi_\Gamma(\lambda)$  is a polynomial of degree  $n$  of the form,

$$\chi_\Gamma(\lambda) = \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} - \dots \pm a_{c(\Gamma)}\lambda^{c(\Gamma)}$$

where  $a_i > 0$  and  $a_1 = |E|$ .

*Proof.* This is a proof by induction. [EXPLAIN HOW.]  $\square$

**Proposition J.4.** [[LABEL P:0912 gen chrom poly]]

$$\chi_\Gamma(\lambda) = \sum_{S \subseteq E} (-1)^{|S|} \lambda^{c(S)}.$$

*Proof.* This proof follows from Lemma J.1 and induction. [BETTER TO EXPLAIN HOW IT FOLLOWS.]  $\square$

## J.2. The Dichromatic Polynomial.

The dichromatic polynomial generalizes the chromatic polynomial to two variables.

**Definition J.1.** [[LABEL D:0912 dichrom poly]] THE DICHROMATIC POLYNOMIAL OF A GRAPH IS DEFINED AS

$$Q_\Gamma(u, v) = \sum_{S \subseteq E} u^{c(S)} v^{[|S| - n + c(S)]}.$$

Notice that  $\chi_\Gamma(\lambda) = (-1)^n Q_\Gamma(-\lambda, -1)$ .

**Lemma J.5.** [[LABEL L:0912 dichrom dc]]  $Q_\Gamma(u, v) = Q_{\Gamma \setminus e}(u, v) + Q_{\Gamma/e}(u, v)$ .

*Proof.* [HAVE WORDS. EXPLAIN WHY THE NOT-OBVIOUS STEPS WORK. THERE WAS A LOT OF DISCUSSION OF IT, AND OF WHY IT DID OR DIDN'T APPLY TO

A LOOP OR ISTHMUS.]

$$\begin{aligned}
Q_\Gamma(u, v) &= \sum_{S \subseteq E} u^{c(S)} v^{|S| - n + c(S)} \\
&= \sum_{S \subseteq E \setminus e} u^{c(S)} v^{|S| - n + c(S)} + \sum_{e \cup S \subseteq E} u^{c(e \cup S)} v^{|e \cup S| - n + c(e \cup S)} \\
&= Q_{\Gamma \setminus e}(u, v) + \sum_{T \subseteq E/e} u^{c_{\Gamma/e}(T)} v^{|T| - |V(\Gamma/e)| + c_{\Gamma/e}(T)} \\
&= Q_{\Gamma \setminus e}(u, v) + Q_{\Gamma/e}(u, v).
\end{aligned}$$

□

[17 September 2008: Peter Cohen]

[Peter] NOTE: will add in graphs soon! The source code has notes where the graphs and diagrams will go in.

**Proposition J.6** (B). [[LABEL P:0917B]]  $Q_\Gamma = Q_{\Gamma \setminus e} + Q_{\Gamma/e}$  for  $e$  not a loop.

*Proof.* The proof is similar to that of Proposition J.7, following. □

**Proposition J.7** (BR). [[LABEL P:0917BR]]  $R_\Gamma = R_{\Gamma \setminus e} + R_{\Gamma/e}$  for  $e$  not a loop or isthmus.

*Proof.* For disjoint  $\Gamma_1$  and  $\Gamma_2$ ,

$$\begin{aligned}
R_\Gamma &= \sum_{S \subseteq E} u^{c(S) - c(\Gamma)} v^{|S| - n + c(S)} \\
&= \sum_{S \subseteq E \setminus e} u^{c(S) - c(\Gamma)} v^{|S| - n + c(S)} + \sum_{e \in S \subseteq E} u^{c(S) - c(\Gamma)} v^{|S| - n + c(S)} \\
&= R_{\Gamma \setminus e} + R_{\Gamma/e}.
\end{aligned}$$

□

**Proposition J.8** (C). [[LABEL P:0917C]]  $Q_{\Gamma_1 \cup \Gamma_2} = Q_{\Gamma_1} Q_{\Gamma_2}$ .

The *vertex amalgamation* of two graphs is defined to be

$$\Gamma_1 \cup_v \Gamma_2 := \Gamma_1 \cup \Gamma_2,$$

where  $\Gamma_1$  and  $\Gamma_2$  share a vertex  $v$  and have no other vertex or edge in common. This frequently occurs, e.g. when isthmi are contracted.

**Proposition J.9** (CR). [[LABEL P:0917CR]]  $R_{\Gamma_1 \cup \Gamma_2} = R_{\Gamma_1 \cup_v \Gamma_2} = R_{\Gamma_1} R_{\Gamma_2}$ .

*Proof.* Consider the case of a vertex amalgamation,  $\Gamma = \Gamma_1 \cup_v \Gamma_2$ . Then

$$\begin{aligned}
R_\Gamma &= \sum_{S \subseteq E_1 \cup E_2} u^{c(S) - c(\Gamma)} v^{|S| - n + c(S)} \\
&= \sum_{S_1 \subseteq E_1} \sum_{S_2 \subseteq E_2} u^{c(S_1 \cup S_2) - c(\Gamma_1 \cup \Gamma_2)} v^{|S_1| + |S_2| - n_1 - n_2 - c(\Gamma_1 \cup \Gamma_2)}
\end{aligned}$$

and noting that  $n = n_1 + n_2 - 1$  and  $c(\Gamma) = c(\Gamma_1) + c(\Gamma_2) - 1$ ,

$$= R_{\Gamma_1} R_{\Gamma_2}.$$

□



[Sept. 15, 2008: Yash Lodha]

The dichromatic polynomial is defined as follows:

$$Q_{\Gamma}(u, v) = \sum_{S \subseteq E} u^{c(S)} v^{|S| - n + c(S)}.$$

The *corank* of  $S \subseteq E$  is defined as  $c(S) - c(\Gamma)$  and its *nullity* is defined as  $|S| - n + c(S)$ . Since  $c(S)$  is obviously at least as large as  $c(\Gamma)$ , and  $n - c(S) = |T| \leq |S|$  for a maximal forest  $T \subseteq S$ , both the corank and nullity are nonnegative. The definitions motivate the name of the following polynomial, called the *rank generating polynomial* or *corank-nullity polynomial*, which is

$$R_{\Gamma}(u, v) := \sum_{S \subseteq E} u^{c(S) - c(\Gamma)} v^{|S| - n + c(S)} = u^{-c(\Gamma)} Q_{\Gamma}(u, v).$$

**The number of spanning trees or maximal forests.**

[September 12: Simon Joyce]

Given a graph  $\Gamma$ , we define  $t(\Gamma)$  to be the number of spanning trees of  $\Gamma$ .

**Lemma J.10.** [[LABEL L:0912 tree dc]]  $t(\Gamma) = t(\Gamma \setminus e) + t(\Gamma/e)$ .

*[WAS THERE A PROOF? Could one be based on the proof for forests given next time? Is there a simple direct proof?]*

[Sept. 15, 2008: Yash Lodha]

Let  $f(\Gamma)$  be the number of maximal forests of  $\Gamma$  and  $t(\Gamma)$  the number of spanning trees. To understand the polynomials discussed above, we calculate them for the graphs  $\emptyset, K_1, K_2, \bar{K}_2$  and compare them with the values of the functions  $f$  and  $t$  for these graphs.

$\Gamma$	$Q_{\Gamma}(u, v)$	$R_{\Gamma}(u, v)$	$t(\Gamma)$	$f(\Gamma)$
$\emptyset$	1	1	0	1
$K_1$	$u$	1	1	1
$K_2$	$u^2 + u$	$u + 1$	1	1
$\bar{K}_2$	$Q_{K_1}(u, v)^2 = u^2$	1	0	1

**Lemma J.11.** [[LABEL L:0915F]] *The following equations hold for the number of maximal forests of a graph.*

$$\begin{aligned} f(\Gamma_1 \cup \Gamma_2) &= f(\Gamma_1)f(\Gamma_2), \\ f(\Gamma) &= f(\Gamma \setminus e) + f(\Gamma/e) \end{aligned}$$

if  $e$  is not a loop or an isthmus, and

$$f(\emptyset) = 1.$$

*Proof.* The first equation follows simply from the fact that the maximal forests of  $\Gamma_1 \cup \Gamma_2$  are in bijective correspondence with pairs of maximal forests in  $\Gamma_1$  and  $\Gamma_2$ . So the result follows from the multiplication principle.

For the second result, we assume that  $e$  is a link. There are two kinds of maximal forest of  $\Gamma$ , the ones that contain  $e$  and the ones that do not contain  $e$ . The ones that contain  $e$  are in bijection with the maximal forests of  $\Gamma/e$  and the ones that do not contain  $e$  are in bijection with the maximal forests of  $\Gamma \setminus e$ . This proves the second equation.

For the third equation, we need only keep in mind the empty forest!  $\square$

**Theorem J.12.** [[LABEL T:0915F]]  $f(\Gamma) = R_\Gamma(0, 0)$ .

*Proof.* Initially, we assume that  $\Gamma$  is connected. We proceed by induction on  $|E|$ . There are three cases—not mutually exclusive.

*Case I:*  $\Gamma$  has a loop  $e$ . Then  $\Gamma = (\Gamma \setminus e) \cup_v K_1^\circ$ . So by Proposition L.7 we get  $R_\Gamma = (1 + v)R_{\Gamma \setminus e}$ . So

$$R_\Gamma(0, 0) = 1 \cdot R_{\Gamma \setminus e}(0, 0) = 1 \cdot f(\Gamma \setminus e) = f(\Gamma)$$

since  $e$  is a loop.

*Case II:*  $\Gamma$  has no loop and every edge is an isthmus. Then  $\Gamma$  is a tree. By inspection we can see that  $f(\Gamma) = 1 = R_\Gamma(0, 0)$ .

*Case III:*  $\Gamma$  has a circle  $C$  of length greater than one. Let  $e \in C$ . Then  $e$  is not a loop or isthmus, therefore by Proposition L.5

$$R_\Gamma(0, 0) = R_{\Gamma \setminus e}(0, 0) + R_{\Gamma/e}(0, 0) = f(\Gamma \setminus e) + f(\Gamma/e) = f(\Gamma).$$

This proves the theorem for the case when  $\Gamma$  is connected.

If  $\Gamma$  has more than one component, we proceed by induction on the number of components of  $\Gamma$ . Let  $\Gamma = \Gamma_1 \cup \Gamma_2$ , where our theorem holds for  $\Gamma_1$  and  $\Gamma_2$  is a connected graph. Then by Propositions L.5 and L.6 and Lemma J.11 we get our inductive step:

$$R_\Gamma(0, 0) = R_{\Gamma_1}(0, 0)R_{\Gamma_2}(0, 0) = f(\Gamma_1)f(\Gamma_2) = f(\Gamma). \quad \square$$

Here are a few examples that illustrate the theorem.

$$\begin{aligned} R_\emptyset(0, 0) &= 1 = f(\emptyset), \\ R_{K_1}(0, 0) &= 1 = f(K_1), \\ R_{K_1^\circ}(0, 0) &= 1 = f(K_1^\circ), \\ R_{K_2}(0, 0) &= 1 = f(K_2). \end{aligned}$$

### J.3. Acyclic Orientations and Proper and Compatible Pairs.

We define  $\text{AO}(\Gamma)$  to be the set of acyclic orientations of  $\Gamma$ . In an oriented graph, the notation  $\vec{P}:v\vec{w}$  means a directed path from  $v$  to  $w$ .

**Lemma J.13.** [[LABEL L:0917ao-dc]] *Consider an orientation  $\vec{\Gamma}$  of  $\Gamma$  and an edge  $e:v_1\vec{v}_2$  in  $\vec{\Gamma}$ .*

- (1) *If there exists  $\vec{P}:v_1\vec{v}_2$  in  $\vec{\Gamma} \setminus e$  then the orientation is cyclic.*
- (2) *If there does not exist  $\vec{P}:v_1\vec{v}_2$  in  $\vec{\Gamma} \setminus e$  then the orientation is acyclic.*

*Proof.* (needs simple graphs to show, will add soon)

*[ARE THESE PROOF CASES RIGHT?  $\vec{P}$  can't be a cycle because  $P$  is not a circle. Or, do you mean it's a cycle in the contraction?]*

*Case 1:  $\vec{\Gamma}/e$  is oriented as in  $\vec{\Gamma} \setminus e$ . In this case,  $\vec{P}$  is not a cycle. [PROOF NEEDED.]*

*Case 2:  $\vec{P}$  doesn't exist, so  $\vec{\Gamma}/e$  is acyclic. [PROOF NEEDED.]*

□

Let  $a(\Gamma)$  denote the number of acyclic orientations of  $\Gamma$ . There is a deletion-contraction formula for this number.

**Lemma J.14.**  $a(\Gamma) = a(\Gamma \setminus e) - a(\Gamma/e)$  if  $e$  is not a loop.

*[MUST DISCUSS. CAN YOU GET ADDITIONAL NOTES FROM JACKIE OR NATE OR SIMON?]*

*(Notes end, need clarification)*

*We will denote a color set,  $(\alpha, \gamma)$ , by  $[K]$ .*

*For two vertices,  $v_1$  and  $v_2$ ,  $\gamma(v_1) \leq \gamma(v_2)$*

[19 Sept.: Simon Joyce]

**Lemma J.15.** [[LABEL L:0919 AO]] *Given a graph  $\Gamma$  and  $e \in E(\Gamma)$  a link then,*

$$\text{AO}(\Gamma) \cup \text{AO}(\Gamma/e) \leftrightarrow \text{AO}(\Gamma \setminus e).$$

*Proof.* If  $\alpha_0 \in \text{AO}(\Gamma \setminus e)$  such that  $\alpha_0$  is also acyclic orientation of  $\Gamma/e$  then  $\alpha_0$  can be extended to an acyclic orientation of  $\Gamma$  by adding  $e:v\vec{w}$  or  $e:\vec{w}v$ . If  $\alpha_0$  is not an acyclic orientation of  $\Gamma/e$  then only one of these is a valid extension to  $\Gamma$ . □

**Definition J.2.** [[LABEL D:0919 c pairs]] *Given an graph  $\Gamma$  and a  $k$ -coloring  $\gamma$  of  $\Gamma$  let*

$$p_\Gamma(k) = \# \text{ of compatible pairs with } k\text{-coloring } \gamma.$$

**Lemma J.16.** [[LABEL L:0919 c pairs DC]] *Given a graph  $\Gamma$  and  $e \in E(\Gamma)$  a link then,*

$$p_{\Gamma \setminus e}(k) = p_\Gamma(k) + p_{\Gamma/e}(k).$$

*Proof.* Fix  $k$  and  $\alpha_0 \in \text{AO}(\Gamma \setminus e)$ . We will prove there exists a 1:1/2:2 correspondence between  $\text{CP}(\Gamma) \cup \text{CP}(\Gamma/e)$  and  $\text{CP}(\Gamma \setminus e)$ .

First we assume  $\alpha_0$  orients  $\Gamma \setminus e$  and  $\Gamma/e$ . This means there is no directed path from  $v$  to  $w$  where  $v$  and  $w$  are the endpoints of  $e$ . We have  $\gamma$  a  $k$ -coloring of  $\Gamma$  that is compatible with  $\alpha_0$  so  $(\alpha_0, \gamma) \in \text{CP}(\Gamma \setminus e)$ .

We have either  $\gamma(v) = \gamma(w)$  or not. In the first case  $\gamma$  colors  $\Gamma/e$  and  $\gamma$  is compatible with both  $e:\vec{v}w$  and  $e:\vec{w}v$ .

In the second case  $\gamma$  doesn't orient  $\Gamma/e$  and  $\gamma$  is compatible with exactly one extension of  $\alpha_0$  since  $\gamma(v) < \gamma(w)$  or visa versa.

Now if there exists an oriented path between  $v$  and  $w$  then without loss of generality assume  $\gamma(v) \leq \gamma(w)$ . Then  $\alpha_0$  extends only by  $e:\vec{v}w$  and since  $\gamma(v) \leq \gamma(w)$  this extension is unique. Calling this extension  $\alpha$  we have  $(\alpha_0, \gamma) \leftrightarrow (\alpha, \gamma)$ .  $\square$

We are now ready to prove our main result.

**Theorem J.17.** [[LABEL T:0919 Stanley's]] *Given a graph  $\Gamma$ ,  $\alpha \in AO(\Gamma)$  and  $\gamma$  a  $k$ -coloring of  $\Gamma$  then,*

$$(-1)^n \chi_\Gamma(-k) = p_\Gamma(k).$$

*Proof.* If  $\Gamma$  has no links then,

$$(-1)^n \chi_\Gamma(-k) = \begin{cases} 0 & \text{If } \Gamma \text{ contains a loop.} \\ (-1)^n (-k)^n & \text{otherwise.} \end{cases}$$

Also

$$p_\Gamma(k) = \begin{cases} 0 & \text{if } \Gamma \text{ contains a loop.} \\ k^n & \text{otherwise.} \end{cases}$$

So in this case we have equality.

If  $\Gamma$  contains a link then we use lemma J.16, deletion-contraction of  $\chi_\Gamma$  and induction.  $\square$

Going back to the idea of coloration, if we take  $\gamma$  to be a  $k$ -coloring of a graph  $\Gamma$  we have  $\gamma : V \rightarrow [k] \subseteq \mathbb{R}$ , so we can think of  $\gamma \in [k]^n \subseteq \mathbb{R}^n$ . So if  $\gamma_i$  and  $\gamma_j$  are the  $i$ th and  $j$ th coordinates of  $\gamma$  then  $\gamma_i \neq \gamma_j$  if  $\exists e_{ij} \in E(\Gamma)$ , i.e.,  $\gamma \notin h_{ij} = \{x : x_i = x_j\}$  for every  $e_{ij} \in E(\Gamma)$ , i.e.,  $\gamma \notin \bigcup \mathcal{H}[\Gamma]$ . So we can redefine a proper coloring as

$$\gamma \in \mathbb{Z}^n \setminus \bigcup \mathcal{H}[\Gamma] \text{ such that } \gamma \in (0, k+1)^n.$$

This can be restated as

$$\frac{\gamma}{k+1} \in \frac{1}{k+1} \mathbb{Z}^n \text{ and } \frac{\gamma}{k+1} \in (0, 1)^n \setminus \bigcup \mathcal{H}[\Gamma].$$

The number of these points is given by the *open Ehrhart polynomial* of  $([0, 1]^n, \mathcal{H}[\Gamma])$ ,  $E^\circ(k+1)$ .

So if we have  $\gamma : V \rightarrow \{0, 1, \dots, k-1\}$  and  $\gamma \in \mathbb{Z}^n \cap [0, k-1]^n$  for  $\alpha \in AO(\Gamma)$  we have  $\alpha$  corresponds to a region  $R(\alpha)$  (defined by  $x_i < x_j$  when  $\exists \vec{v}_i \vec{v}_j$  in  $\alpha$ ) of  $\mathcal{H}[\Gamma]$ . Also we have  $\gamma_i \leq \gamma_j$  when  $\exists \vec{v}_i \vec{v}_j$  in  $\alpha$ . This defines the *closure*  $\bar{R}(\alpha)$  of  $R(\alpha)$  called the *closed region* of  $\alpha$ . So given  $\gamma$  the number of compatible pairs  $(\alpha, \gamma) = \#$  of closed regions of  $\mathcal{H}[\Gamma]$  that contain  $\gamma$ .

**Definition J.3.** [[LABEL D:0919 C Ehr poly]] Given a graph  $\Gamma$  and  $x \in \mathbb{R}^n$  we define

$$m(x) := \text{number of closed regions of } \mathcal{H}[\Gamma] \text{ that contain } x.$$

Now we define the *closed Ehrhart polynomial* to be

$$E(k-1) = \sum_{\gamma \in \mathbb{Z}^n \cap [0, k-1]^n} m(\gamma).$$

So we have (by some calculation)  $E^0(t) = (-1)E(-t)$  for  $[0, 1]^n$  and  $\mathcal{H}[\Gamma]$  from Stanley's theorem.

[September 24, 2008: Nate Reff]

#### J.4. The Tutte polynomial.

The Tutte polynomial is a universal function that satisfies the relations we've been discovering for the corank-nullity polynomial and other polynomials. Let's review these relations. We found:

- Deletion-Contraction Property:  
 $Q_\Gamma = Q_{\Gamma \setminus e} + Q_{\Gamma/e}$  if  $e$  is not a loop.  
 $R_\Gamma = R_{\Gamma \setminus e} + R_{\Gamma/e}$  if  $e$  is not a loop or isthmus.
- Disjoint Graph Multiplicativity:  
 $Q_{\Gamma_1 \cup \Gamma_2} = Q_{\Gamma_1} Q_{\Gamma_2}$  and  $R_{\Gamma_1 \cup \Gamma_2} = R_{\Gamma_1} R_{\Gamma_2}$ .
- Multiplicativity:  
 Disjoint Graph Multiplicativity, and  $R_{\Gamma_1 \cup_v \Gamma_2} = R_{\Gamma_1} R_{\Gamma_2}$ .
- Empty-Graph Unitarity:  
 $Q_\emptyset = 1 = R_\emptyset$ .
- Unitarity:  
 Empty-Graph Unitarity, and  $R_{K_1} = 1$ .
- Invariance:  
 $\Gamma_1 \cong \Gamma_2 \implies Q_{\Gamma_1} = Q_{\Gamma_2}$  and  $R_{\Gamma_1} = R_{\Gamma_2}$ .

We call a *Tutte–Grothendieck invariant of graphs* any function  $F$  on graphs that satisfies all these properties. Let's restate them precisely, in the generality of an arbitrary function  $F$  defined on graphs:

- (DC) **Deletion-Contraction Identity:**  
 $F(\Gamma) = F(\Gamma \setminus e) + F(\Gamma/e)$  if  $e$  is not a loop or isthmus.
- (M) **Multiplicativity:**  
 $F(\Gamma_1 \cup \Gamma_2) = F(\Gamma_1 \cup_v \Gamma_2) = F(\Gamma_1)F(\Gamma_2)$ .
- (U) **Unitarity:**  
 $F(\emptyset) = F(K_1) = 1$ .
- (I) **Invariance:**  
 $\Gamma_1 \cong \Gamma_2 \implies F(\Gamma_1) = F(\Gamma_2)$ .

Now let's look at what it means for a function to satisfy these properties, and head toward answering Tutte's question of what are all such functions. First of all, in order for all the properties to make sense,  $F$  has to have values in a commutative ring with unity. Next, because of the multiplicativity property (M),  $F(\Gamma) =$  the product of  $F(\text{blocks})$ . Due to the property of invariance (I),  $F(\text{loop}) =$  a value  $y$  that is the same for all loops, and also  $F(\text{isthmus}) =$  a value  $x$  that is the same for all isthmi. Due to unitarity and multiplicativity,  $F(\Gamma) = 1$  if  $\Gamma$  has no edges. And lastly, there is the following special case:

**Lemma J.18.** [[LABEL L:0924lemma1 Loop Isthmus Lemma]] *Suppose  $\Gamma$  has  $l$  loops and  $i$  isthmi and no other edges. Then  $F(\Gamma) = x^i y^l$ .*

As another side comment we note that, if the codomain of  $F$  is an integral domain, then (U) is almost superfluous; that is, it can be deduced from the other properties, except for a small number of functions  $F$ . (This is left as a homework exercise. Hint: Derive (U) from (M) and (I); find the exceptional cases.)

Returning to J.18, let's look at some simple examples. Suppose we define a graph  $G$  as in figure J.1. Let  $C_2$  denote the digon graph, which is a circle of length 2; it consists of two vertices and two parallel edges between those vertices. Now for the calculation of  $F$  using the deletion-contraction method (as seen in figure J.1) we get the following:

$$\begin{aligned}
 F(G) &= F(G \setminus e) + F(G/e) \\
 &= F((G \setminus e) \setminus a) + F((G \setminus e)/a) + F((G/e) \setminus a) + F((G/e)/a) \\
 &= (x^3) + F(K_3) + xF(C_2) + yF(C_2) \\
 &= (x^3) + (x^2 + x + y) + x(x + y) + y(x + y) \\
 &= x^3 + 2x^2 + x + 2xy + y + y^2.
 \end{aligned}$$

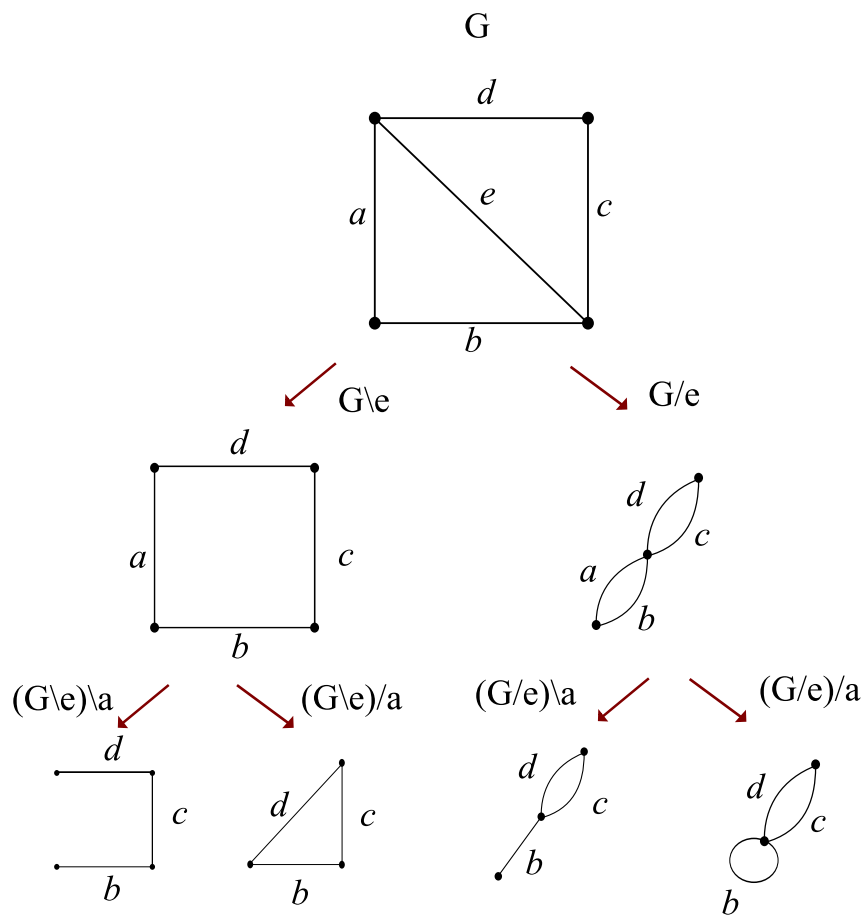


FIGURE J.1. Calculation of  $F$  by the method of deletion and contraction.  
 [[LABEL F:0924Figure1]]

**Theorem J.19.** [[LABEL T:0924Theorem1 Main Theorem]] *Suppose  $F$  is a Tutte–Grothendieck invariant of graphs. Let  $x = F(\text{isthmus})$ , and let  $y = F(\text{loop})$ . Then*

- (1)  $F(\Gamma) = R_\Gamma(x - 1, y - 1)$ , a polynomial function of  $x$  and  $y$ ,
- (2) the polynomial has nonnegative integral coefficients, and
- (3) any evaluation of  $R_\Gamma(x, y)$  gives a Tutte-Grothendieck invariant of graphs.

*Proof.* One proves the first two statements by induction on  $|E|$ , using (DC) and (M). The third statement follows from the fact that  $R_\Gamma$  itself is a Tutte-Grothendieck invariant.  $\square$

**Corollary J.20.** [[LABEL T:0924Corollary1 Main Corollary]] *A Tutte–Grothendieck invariant  $F$  is well defined given any choices of  $x = F(\text{isthmus})$  and  $y = F(\text{loop})$  and is uniquely determined by those choices.*

*Proof.* This is an immediate corollary of Theorem J.19.  $\square$

*The Tutte polynomial.*

We define the *Tutte polynomial* of  $\Gamma$  as the polynomial obtained by reducing a general  $F(\Gamma)$  to  $x$ 's and  $y$ 's using the properties defining a Tutte-Grothendieck invariant of graphs. We denote the Tutte polynomial by  $T_\Gamma(x, y)$ . Our main theorem, Theorem J.19, tells us that  $T_\Gamma(x, y) = R_\Gamma(x - 1, y - 1)$ . Using previous results we can now also write  $T_\Gamma(1, 1) = R_\Gamma(0, 0) = f(\Gamma)$ , and  $T_\Gamma(1 - \lambda, 0) = R_\Gamma(-\lambda, -1) = (-1)^n \chi_\Gamma(\lambda)$  as well as many other such forms.

**Theorem J.21.** [[LABEL T:0924Theorem2]]  *$T_\Gamma(x, y)$  is a polynomial, with no constant term if  $|E| > 0$ . The degree of  $x$  equals  $\text{rk}(\Gamma) = n - c(\Gamma)$ , and the degree of  $y$  equals the nullity of  $\Gamma$ , that is,  $|E| - n + c(\Gamma)$ . Furthermore,  $b_{ij} = 0$  if  $i + j >$  the cyclomatic number (nullity) of  $\Gamma$ .*

*Proof.* This is an immediate corollary of Theorem J.19.  
[NOT SO IMMEDIATE. Needs some indication of proof.]  $\square$

Let's take another look at the subset expansion of the corank-nullity polynomial:

$$(J.1) \quad R_\Gamma(u, v) = \sum_S u^{c(S) - c(\Gamma)} v^{|S| - n + c(\Gamma)} = \sum_{k,l} a_{kl} u^k v^l, \text{ [[LABELE : 0924Tutte1]]}$$

where  $a_{kl}$  is the coefficient of  $u^k v^l$ , that is, the number of subsets  $S \subseteq E$  that have rank  $k = c(S) - c(\Gamma)$  and nullity  $l = |S| - n + c(S)$ . Write

$$T_\Gamma(x, y) = \sum_{i,j \geq 0} b_{ij} x^i y^j.$$

Then all  $b_{ij} \geq 0$ ; this can be proved by induction. We deduce that

$$\begin{aligned}
R_\Gamma(u, v) &= T_\Gamma(u + 1, v + 1) \\
&= \sum_{i, j \geq 0} b_{ij} (u + 1)^i (v + 1)^j \\
&= \sum_{i, j \geq 0} b_{ij} \sum_k \binom{i}{k} u^k \sum_l \binom{j}{l} v^l \\
&= \sum_{i, l \geq 0} u^k v^k \underbrace{\sum_{i, j \geq 0} b_{ij} \binom{i}{k} \binom{j}{l}}_{\text{coefficients of } u^k v^k \text{ in (J.1)}} \\
&= \sum_{k, l} a_{kl} u^k v^l.
\end{aligned}$$

This string of equalities shows that  $a_{kl} = \sum_{i, j \geq 0} b_{ij} \binom{i}{k} \binom{j}{l}$ . This allows us to get good lower bounds for certain graph quantities by looking at the coefficients of the Tutte polynomial. We infer, not only that  $a_{kl} \geq 0$ , but stronger positivity due to the fact that  $a_{kl}$  is a positive combination of nonnegative integers  $b_{ij}$ .

Here are some significant properties of the Tutte polynomial, that we will not prove. A graph is said to be *separable* if it is not 2-connected or it has a loop. A *series-parallel graph* is a graph such that each block is derived from a single edge by repeatedly subdividing edges and adding parallel edges. Assuming  $|E(\Gamma)| \geq 2$ , we can say that:

- $b_{01} = b_{10}$ .
- $b_{01} = 0 \iff \Gamma$  is separable.
- $b_{01} = 1 \iff \Gamma$  is a series-parallel graph.

### J.5. Coefficients of the chromatic polynomial.

Now let's take a look at the chromatic polynomial. We define  $w_i$ , called the *Whitney numbers of the first kind* of  $\Gamma$ , to be the coefficients of powers of  $\lambda$  in the chromatic polynomial:  $\chi_\Gamma(\lambda) = \sum_{i=0}^n w_i \lambda^{n-i}$ . Now we can say that:

$$\begin{aligned}
\sum_{k=0}^n (-1)^{n-k} w_{n-k} \lambda^k &= (-1)^n \chi_\Gamma(-\lambda) \\
&= T_\Gamma(1 + \lambda, 0) \\
&= Q_\Gamma(\lambda, -1) \\
&= \sum_{i, j \geq 0} (1 + \lambda)^i 0^j b_{ij} \\
&= \sum_i (1 + \lambda)^i b_{i0} \\
&= \sum_k \lambda^k \sum_i \binom{i}{k} b_{i0}.
\end{aligned}$$

Therefore,  $w_{n-k} = (-1)^{n-k} \sum_i \binom{i}{k} b_{i0}$ . The sum is nonnegative; thus we have the following theorem.



**Theorem J.22.** [[LABEL T:0924Theorem3 Alternating Sign Theorem]] *The Whitney numbers  $w_i$  alternate in sign, with  $w_0 = 1$  and  $(-1)^i w_i \geq 0$ .*

This tells us that the coefficients of the chromatic polynomial alternate in sign. More can be said about the Whitney numbers with further study involving the Tutte polynomial, but that belongs to matroid theory and would take us too far afield.

## K. LINE GRAPHS

[09/26/08: Yash Lodha]

The *line graph* of  $\Gamma$ , denoted by  $L(\Gamma)$ , is defined as follows:

$$\begin{aligned} V(L(\Gamma)) &= E(\Gamma), \\ E(L(\Gamma)) &= \{ef \mid e, f \text{ are adjacent in } \Gamma\}. \end{aligned}$$

(Recall that edges are *adjacent* when they have a common vertex.) This is the simple definition, valid for simple graphs  $\Gamma$ .

The definition of line graphs raises a few important questions regarding them. First of all, which graphs are line graphs? Secondly, are there graphs that are isomorphic to their line graphs? Thirdly, how many non isomorphic graphs can produce the same line graph? We now provide a few examples:

- (1)  $L(K_3) \cong K_3$ .
- (2)  $L(K_3) \cong K_3$ .

According to a theorem of Whitney's, these are the only two connected (simple) graphs that have the same line graph.

[I then go on to describe graphically what happens with double edges and loops with graphics.] [NEEDED!]

Let  $\Gamma$  be a simple graph. Let  $B$  be the unoriented incidence matrix of  $\Gamma$ , and let  $H$  be the oriented matrix of  $\Gamma$ . Then the entry  $x_{i,j}, i \neq j$ , of  $BB^T$  is the number of  $ij$  edges for vertices  $i, j \in V(\Gamma)$  and entry  $x_{i,i}$  of  $BB^T$  is the degree valency of the vertex  $i$ . It is clear from this that  $BB^T = D + A$  where  $D$  is the degree matrix or the diagonal  $V \times V$  matrix with degrees of vertices in its diagonal entries and  $A$  is the adjacency matrix. The entry  $x_{i,j}, i \neq j$  of  $HH^T$  is negative of the number of  $vw$  edges, and the entry  $x_{i,i}$  of  $HH^T$  is the degree of the vertex  $i$ .

**Theorem K.1.** [[LABEL T:0926rge]] *If  $\Gamma$  is loopless and  $k$ -regular, then the largest eigenvalue of  $A$  is  $k$ , with multiplicity at least  $c(\Gamma)$ .*

The actual multiplicity is exactly  $c(\Gamma)$ , but I won't prove it.

*Proof.* Notice that  $HH^T$  is a *Gram matrix* (which is defined as a matrix  $G$  of inner products of vectors in  $\mathbb{R}^n$ , i.e., where  $g_{i,j} = v_i \cdot v_j$ , the dot product of vectors  $v_i, v_j$ ). This is positive semidefinite, which means that it is symmetric and  $\forall x \in Y, Ax \cdot x \geq 0$ . So all eigenvalues are greater than or equal to zero.

Let  $x$  be an eigenvector of  $A$  with eigenvalue  $\lambda$ . Then  $Ax = \lambda x$ . And  $HH^T x = kIx - Ax = (k - \lambda)x$ . This implies that  $x$  is an eigenvector of  $HH^T$  with eigenvalue  $k - \lambda$ .

To show  $k$  is an eigenvalue with multiplicity greater than or equal to  $c(\Gamma)$ , suppose the components have vertex sets  $V_1 = \{v_1, \dots, v_{n_1}\}$ ,  $V_2 = \{v_{n_1+1}, \dots, v_{n_1+n_2}\}$ ,  $\dots$ . So  $\pi(\Gamma) = \{V_1, V_2, \dots, V_{c(\Gamma)}\}$ . Let  $x_i \in \mathbb{R}^n$  be the vector which is 0 except for being 1 on every vertex of  $V_i$ . It is easy to see that  $Ax_i = kx_i$ . Therefore we have at least  $c(\Gamma)$  independent eigenvectors, hence  $k$  has multiplicity at least  $c(\Gamma)$ .  $\square$

Now we look at  $B^T B$ , which is an  $E \times E$  matrix. In this matrix the entry  $x_{i,j}$  is the number of edges between the vertices  $v_i, v_j$ , and  $x_{i,i}$  is the degree of vertex  $v_i$ . It is clear that  $B^T B = A(L) + 2I$ , where  $L = L(\Gamma)$ . Since  $H^T H$  is positive semidefinite, the eigenvalues are greater than or equal to zero.

**Theorem K.2.** [[LABEL T:0926lge]] *The eigenvalues of a line graph are greater than or equal to  $-2$ .*

*Proof.* Let  $\lambda$  be an eigenvalue of  $A(L)$  with eigenvector  $x$ . Then  $A(L)x = \lambda x$ . Now

$$B^T Bx = (A(L) + 2I)x = (\lambda + 2)x.$$

This implies that  $\lambda + 2$  is an eigenvalue of  $B^T B$ . So  $\lambda \geq -2$ .  $\square$

## L. CYCLES, CUTS AND THEIR SPACES

[09/26/08: Yash Lodha]

**L.1. Cycles and Cuts.** A *cut* or *cutset* is the set of edges between a vertex set  $X \subseteq V$  and its complement  $X^c$  (if this set is nonempty). We define  $E(X, X^c)$  to be the edge set with one endpoint in  $C_1$  and the other in  $C_2$ . A *bond* is a minimal cut.

**Theorem L.1.** *Every cut is a disjoint union of bonds in a unique way.*

*Proof.* Consider the vertex sets  $X$  and  $X^c$ . Let  $E(X, X^c)$  be the cutset defined above. For  $e \in E(X, X^c)$ , let  $v_1 \in X, v_2 \in X^c$  be the vertices incident on  $e$ . Then consider the component  $C_1$  of the subgraph induced by  $X$  which contains  $v_1$  and the component  $C_2$  of the subgraph induced by  $X^c$  which contains  $v_2$ . Let  $E(C_1, C_2)$  be the edge set with one endpoint in  $C_1$  and the other in  $C_2$ . Now it is clear that  $E(C_1, C_2)$  is a bond, since  $C_1$  and  $C_2$  are connected, removing a proper subset of  $E(C_1, C_2)$  will leave  $C_1 \cup C_2$  connected, and hence not increase the number of components of our graph.

From here it follows that  $E(X, X^c)$  is the unique disjoint union of edge sets (which are bonds) connecting a pair of components of  $X$  and  $X^c$ .  $\square$

[Sept. 15, 2008: Yash Lodha]

## Nullity and cyclomatic number.

**Proposition L.2.** [[LABEL P:0915cyclo]] *The cyclomatic number of the subgraph  $(V, S)$  induced by  $S \subseteq E$  is  $|S| - n + c(S)$ .*

*Proof.* For each component  $C$  of  $S$ , the cyclomatic number of  $C$  is  $|E(C)| - |V(C)| + 1$ . The cyclomatic number of  $S$  is this summed over all components, i.e.,  $\sum_C (|E(C)| - |V(C)| + 1) = |S| - n + c(S)$ .  $\square$

**Corollary L.3.** [[LABEL C:0915nul]] *The nullity of  $\Gamma$  equals the cyclomatic number of  $\Gamma$ , which equals the number of independent circles.*

To explain independence of circles we need the binary cycle space.

### The binary cycle space.

Given a maximal forest  $T$  of  $\Gamma$ , if we add another edge  $e$  we obtain a circle. This circle is called the *fundamental circle* associated with  $e$ , written  $C_T(e)$ . The entire set  $\{C_T(e) \mid e \notin T\}$  is called the *fundamental system of circles* associated with  $T$ .

**Proposition L.4.** [[LABEL P:0915fundbasis]] *Given  $T$ , every circle is a set sum of fundamental circles in a unique way.*

Under set summation  $\mathcal{P}(E)$  is a binary vector space. The *binary cycle space* is the subspace spanned by all circles. It is not difficult to see, using the theorem above, that any fundamental system of circles is a basis of the binary cycle space.

**Proposition L.5.** [[LABEL P:0915BR]]  *$R_\Gamma = R_{\Gamma \setminus e} + R_{\Gamma/e}$  if  $e$  is not a loop or an isthmus.*

We write  $S/e$  to mean the set  $S \setminus e$  in the graph  $\Gamma/e$ .

*Proof.* We use the facts that  $c(S) = c(S/e)$  for  $e \in S$  and  $c(S) = c(S \setminus e)$  for  $e \notin S$ .

$$\begin{aligned} R_{\Gamma \setminus e} + R_{\Gamma/e} &= \sum_{S \subseteq E \setminus e} u^{c(S) - c(\Gamma \setminus e)} v^{|S| - n + c(S)} + \sum_{e \in S \subseteq E} u^{c(S) - c(\Gamma/e)} v^{|S| - n + c(S)} \\ &= \sum_{S \subseteq E \mid e \notin S} u^{c(S) - c(\Gamma)} v^{|S| - n + c(S)} + \sum_{S \subseteq E \mid e \in S} u^{c(S) - c(\Gamma)} v^{|S| - n + c(S)} \\ &= R_\Gamma. \end{aligned} \quad \square$$

The following propositions follow directly from the multiplication principle. [EH? I don't see it. Some better indication of proof is needed – or just write proofs.]

**Proposition L.6.** [[LABEL P:0915CR]]  $R_{\Gamma_1 \cup \Gamma_2} = R_{\Gamma_1} R_{\Gamma_2}$ .

**Proposition L.7.** [[LABEL P:0915DR]]  $R_{\Gamma_1 \cup_v \Gamma_2} = R_{\Gamma_1} R_{\Gamma_2}$ .

Circuits and Bonds. Cycles (defs. b, c, d) and cocycles..

# CHAPTER II. SIGNED GRAPHS

## A. TECHNICAL DEFINITIONS

o Signed graph. o Walk and circle signs. o Balance. Criteria for balance. o Switching.

## B. BASIC STRUCTURES

**B.1. Signature as a homomorphism.**

**B.2. Balanced circles and the oddity condition.**

## C. CONNECTEDNESS

**C.1. Balanced components.**  $b(\Sigma)$ ,  $\pi_b(\Sigma)$ .

**C.2. Unbalanced blocks.**

## D. MEASUREMENT OF IMBALANCE

**D.1. Frustration index.**

**D.2. Vertex-disjoint negative circles.**

**D.3. Edge-disjoint negative circles.**

## E. MINORS

Deletion and contraction by an edge or edge set.

## F. CLOSURE

Lattice of closed sets and lattice of partial signed partitions.

## G. INCIDENCE AND ADJACENCY MATRICES

## H. ORIENTATION

o Bidirected graphs. o Incidence matrix. o Cycles. Acyclic and totally cyclic orientations.

## I. EQUATIONS AND INEQUALITIES FROM EDGES

Consistency via acyclic orientation.

## J. COLORING

o Proper vs. improper. o Chromatic numbers  $\chi(\Sigma)$  and  $\chi^*(\Sigma)$ .

## K. CHROMATIC FUNCTIONS

o Set of improper edges. o Chromatic polynomials: ordinary, balanced (zero-free), and comprehensive. Deletion-contraction and multiplicativity. Subset expansion. o Acyclic orientations: proper and compatible pairs. o Dichromatic polynomials. Deletion-contraction and multiplicative identities. o Tutte polynomial.

L. LINE GRAPHS

M. CYCLE AND COCYCLE SPACES

Circuits and Cocircuits.

# CHAPTER III. GAIN GRAPHS AND BIASED GRAPHS

## A. TECHNICAL DEFINITIONS

A.1. **Gain Graphs.** ◦ Gain graph. ◦ Walk and circle signs. ◦ Balance. ◦ Switching.

A.2. **Biased Graphs.** ◦ Biased graph. ◦ Balance. Balanced partition.

## B. MINORS

Deletion and contraction by an edge or edge set.

## C. CLOSURE

Lattices of closed sets and of partial partitions. The two matroids. Rank.

## D. INCIDENCE MATRICES OF A GAIN GRAPH

## E. VECTOR REPRESENTATIONS

## F. HYPERPLANE REPRESENTATIONS

multiplicative and additive

## G. COLORING AND CHROMATIC FUNCTIONS

G.1. **Gain Graphs.** ◦ Proper vs. improper. ◦ Chromatic polynomials. Deletion-contraction and multiplicativity. Subset expansion. ◦ Dichromatic polynomials. Deletion-contraction and multiplicative identities.

G.2. **Biased Graphs.** ◦ Chromatic polynomials. Deletion-contraction and multiplicativity. Subset expansion. ◦ Dichromatic polynomials. Deletion-contraction and multiplicative identities. ◦ Tutte polynomial.

# CHAPTER IV. GEOMETRY

## A. LINEAR, AFFINE, AND PROJECTIVE SPACES

over a field or division ring.

## B. VARIOUS GEOMETRICAL REPRESENTATIONS

o Vectors. o Hyperplanes. Regions. o Angles. Root systems.

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