Signed Graphs: Math 581, Spring 2017

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Notes by ...

The notes by the students have been edited by the lecturer.

Contents

Notes for 18 Jan. 2017 – Amelia Mattern.

1. Fundamentals

Everything is finite unless otherwise stated. (That excludes the obvious, like $\mathbb{Z}$ and $\mathbb{R}$.) Infinite graph theory is a topic in itself that is way outside our scope.)

Definition 1. A *graph* Γ is a pair (V, E) in which: V and E are disjoint sets. An element of V is called a *vertex*; an element of E is called an *edge*. Each edge has a multiset of *endpoints*, $V(e) \subseteq V$, consisting of no more than 2 vertices.

A *link* is an edge with 2 distinct endpoints; if its endpoints are v and w, it may be written $e{:}vw$. A *loop* is an edge with 2 coincident endpoints; if its endpoints are v, it may be written $e{:}vv$. A *half edge* is an edge with 1 endpoint; it may be written $e{:}v$.

In $\Gamma = (V, E)$, E^* denotes the set of loops and links.

Definition 2. A *circle* is a connected graph of order greater than 0 with degree (valency) 2 at every vertex. A loose edge is not a connected component.[1] Note that a loop $e{:}vv$ has degree 2, due to its two ends.

Definition 3. A *signed graph*, $\Sigma = (\Gamma, \sigma) = (V, E, \sigma)$, is a graph Γ with a sign function, or *signature*, $\sigma : E^* \to \{+, -\}$.

We may use any of several notations for the sign group: $\{+, -\}$, $\{+1, -1\}$, $\mathbb{Z}_2$, or $\mathbb{F}_2$. The important thing is that each one is a group of order 2.

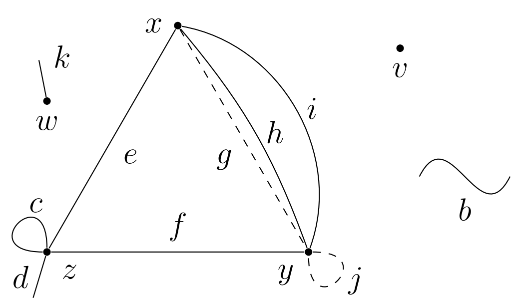

Figure 1.1. The four kinds of edge and various vertices. Solid lines are positive edges (c, e, f, h, i), dashed lines are negative edges (g, j), except that half edges (d, k) and loose edges (b), which have no sign, are solid. The isolated vertex v has degree 0; z has degree 5 since the loop has two ends at z. The xy edges are three parallel edges (they have the same endpoints) but only two multiple edges (i, h; "multiple" edges have the same sign). The positive circles are $\{c\}$, $\{e, f, h\}$, $\{e, f, i\}$, $\{h, i\}$. The negative circles are $\{j\}$, $\{e, f, g\}$, $\{g, h\}$, $\{g, i\}$.

[1] More precisely we could define vertex components and edge components. A loose edge is not a vertex component; an isolated vertex is not an edge component. In these notes we never use edge components; all components are vertex components.

Definition 4. *Switching* a vertex z means negating the sign of each link incident to z. Switching a set X of vertices means negating the sign of every link with one endpoint in X and the other in $X^c := V \setminus X$. The switched graph is $\Sigma^X = (\Gamma, \sigma^X)$.

An equivalent definition of switching uses a *switching function* $\zeta : V \to \{+, -\}$. We define the switched graph to be $\Sigma^\zeta := (\Gamma, \sigma^\zeta)$ where $\sigma^\zeta(e) = \zeta(u)\sigma(e)\zeta(v)$ for an edge $e{:}uv$. The two definitions are equivalent because $\Sigma^\zeta = \Sigma^X$ where $X = \zeta^{-1}(-)$.

The importance of switching is, in part, that it does not change the signs of circles (Theorem 2). In particular, loops do not change sign when incident to a switched vertex since we negate at both ends of the loop. For the full force of switching, see Theorem 2.

Definition 5. For a signed graph $\Sigma = (\Gamma, \sigma)$:

$|\Sigma|$ denotes the underlying graph Γ.
$\mathscr{B}(\Sigma)$ denotes the set of positive circles.
$\Sigma_1 \sim \Sigma_2$ means Σ_1 can be switched to Σ_2; we say Σ_1 is *switching equivalent* to Σ_2.

Switching equivalence is an equivalence relation. This follows from the fact that $(\Sigma^X)^Y = \Sigma^{X \oplus Y}$, where $\oplus$ denotes the symmetric difference of sets. The equivalence class of Σ under switching is denoted by $[\Sigma]$ and is called the *switching class* of Σ.

Lemma 1. *Let F be a maximal forest in Σ. Then Σ can be switched to have any desired signs on F. The resulting signs on $|\Sigma|$ are then uniquely determined.*

Proof. We may assume Σ is connected; then F is a spanning tree. Choose a root vertex r and for each vertex v let F_{rv} denoted the unique path in F between r and v. Define $\zeta(v) := \sigma(F_{rv})$. Then F is all positive in Σ^X. Let σ_F denote the desired signature of F and define $\xi(v) := \sigma_F(F_{rv})$. Then $(\Sigma^\zeta)^\xi$ has the desired signs on F and is switching equivalent to Σ.

The signs are uniquely determined because an edge not in F, $e{:}uv$, has sign $\xi(u)\zeta(u) \cdot \sigma(e) \cdot \zeta(v)\xi(v)$ after switching. □

Theorem 2. *Let Σ_1 and Σ_2 have the same underlying graph. Then $\Sigma_1 \sim \Sigma_2$ if and only if $\mathscr{B}(\Sigma_1) = \mathscr{B}(\Sigma_2)$.*

Proof. Choose a maximal forest F in the underlying graph.

Suppose $\Sigma_2 = \Sigma_1^\zeta$ for some switching function ζ. Consider a circle $C : v_0 e_1 v_1 e_2 \cdots v_{l-1} e_l v_l$, where $l \geq 1$ and $v_0 = v_l$. Then

$$\begin{aligned}
\sigma_2(C) &= \sigma_2(e_1)\sigma_2(e_2)\cdots\sigma_2(e_l) \\
&= [\zeta(v_0)\sigma_1(e_1)\zeta(v_1)][\zeta(v_1)\sigma_1(e_2)\zeta(v_2)]\cdots[\zeta(v_{l-1})\sigma_1(e_l)\zeta(v_l)] \\
&= \zeta(v_0) \cdot \sigma_1(e_1)\sigma_1(e_2)\cdots\sigma_1(e_l) \cdot \zeta(v_l)
\end{aligned}$$

because $\zeta(v)^2 = +$ for every vertex

$$= \sigma_1(C)$$

for the same reason.

Suppose $\mathscr{B}(\Sigma_1) = \mathscr{B}(\Sigma_2)$. Switch Σ_2 by X so that it has the same signs as Σ_1 on F; by the previous part, that does not change the circle signs. Consider an edge $e{:}uv$ not in F and the unique circle $C_e \subseteq F \cup \{e\}$. Then $\sigma_1(e)\sigma_1(F_{uv}) = \sigma_1(C) = \sigma_2(C) = \sigma_2^X(e)\sigma_2^X(F_{uv}) = \sigma_2^X(e)\sigma_1(F_{uv})$, from which we conclude that $\sigma_1(e) = \sigma_2^X(e)$. Thus, $\Sigma_1 = \Sigma_2^X$. □

Corollary 3. *A signed graph is balanced if and only if it switches to an all-positive signed graph without half edges.*

Continuing to introduce fundamental notions of signed graphs:

Definition 6. Two signed graphs Σ_1 and Σ_2 are *isomorphic*, written $\Sigma_1 \cong \Sigma_2$, if there is a sign-preserving graph isomorphism $|\Sigma_1| \cong |\Sigma_2|$.

Two signed graphs Σ_1 and Σ_2 are *switching isomorphic*, written $\Sigma_1 \simeq \Sigma_2$, if there is a switching Σ_1' of Σ_1 such that $\Sigma_1' \cong \Sigma_2$.

Definition 7. Σ is *balanced* if every circle is positive and Σ has no half edges. Σ is called *unbalanced* or (esp. in physics) *frustrated* if it is not balanced.

Definition 8. The *frustration index* $l(\Sigma)$ is $\min\{\#S \mid S \subseteq E \text{ and } \Sigma \setminus S \text{ is balanced}\}$.

The vertex analog is the *frustration number* $l_0(\Sigma)$, defined as $\min\{\#T \mid T \subseteq V \text{ and } \Sigma \setminus T \text{ is balanced}\}$

Notes for 20 Jan. 2017 – Josh Carey.

We begin with some essential definitions from graph theory, just to settle the terminology.[2]

- **Walk::** A sequence of vertices and edges: $v_0e_1v_1 \ldots e_lv_l$ such that $V(e_i)$ is the multiset $\{v_{i-1}, v_i\}$ and $l \geq 0$.
- **Trail::** A walk with no repeated edges.
- **Closed and Open::** A walk or trail is *closed* when $v_0 = v_l$ and *open* when $v_0 \neq v_l$.
- **Path::** A trail with no repeated vertices.
- **Closed Path::** A closed trail where $v_0 = v_1$ is the only repeated vertex. (Note that a closed path is not a path!)

The sign of a walk $W = e_1e_2 \cdots e_l$ is the product of its edge signs, counting edges by the number of times they appear in W; in a formula,

$$\sigma(W) = \prod_{i=1}^{l} \sigma(e_i).$$

The signs of walks, especially closed walks, and most especially closed paths, are very important!

Now, here is the first theorem of signed graph theory. (Literally.)

Theorem 4 (Harary's Balance Theorem [6])**.** *Let Σ be a signed graph. Then the following are equivalent:*

(o) *Σ is balanced.*
(i) *Every circle of Σ is positive.*
(ii) *V has a bipartition, $V = X \sqcup Y$, such that every edge within X or Y is $+$ and every edge between X and Y is $-$.*
(iii) *For every pair of vertices, v and w, every vw-path has the same sign.*

[2] It's graph theory. Up to observational error, no two graph theorists use the same terminology and notation.

The bipartition described in (ii) above is called a *Harary Bipartition*.[3] As an extreme case, if $\Sigma = +\Gamma$ (i.e., all edges are +), then the Harary Bipartition is $\{V, \varnothing\}$.

We now begin to prove Harary's Balance Theorem:

Proof.(ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii) are easy.

For (iii) $\Rightarrow$ (i), let C be a circle and $v, w \in V(C)$. There are two paths, P and Q, joining v, w in C. By (iii), they have the same sign, $\sigma(C) = \sigma(P \cup Q) = \sigma(P) \cdot \sigma(Q) = +$.

Before we continue our proof, we must introduce the concept of switching. Let Σ be a signed graph and $X \subseteq V$. By *switching* X we mean changing Σ to $\Sigma^X = (|\Sigma|, \sigma^X)$ where

$$\sigma^X(e{:}vw) = \begin{cases} \sigma(e{:}vw), & \text{if } v \text{ and } w \text{ are both in } X \text{ or both in } X^c, \\ -\sigma(e{:}vw), & \text{if one of } v, w \text{ is in } X \text{ and the other is in } X^c. \end{cases}.$$

Note that $(\Sigma^X)^Y = \Sigma^{X \oplus Y}$.

There is another way to view switching. A *switching function*, is a function $\zeta : V \to \{+, -\}$. We can switch Σ by ζ to get a new signature: $\Sigma^\zeta = (|\Sigma|, \sigma^\zeta)$ where $\sigma^\zeta(e{:}vw) := \zeta(v)\sigma(e)\zeta(w)$.

To prove the two kinds of switching are equivalent, define

$$\zeta_X(w) = \begin{cases} - \text{ if } w \in X, \\ + \text{ if } w \notin X, \end{cases}$$

then $\Sigma^X = \Sigma^{\zeta_X}$. Conversely, given ζ, let $X = \zeta^{-1}(-)$.

Proposition 5. *Let Σ be a signed graph, C a circle in Σ and P a vw-path in Σ. Let ζ, ξ be switching functions for Σ. Then:*

(i) $(\Sigma^\zeta)^\xi = \Sigma^{\zeta\xi}$.
(ii) $\sigma^\zeta(C) = \sigma(C)$.
(iii) $\sigma^\zeta(P) = \zeta(v)\sigma(P)\zeta(w)$.

Therefore, switching doesn't change properties (i) or (iii).

We now continue our proof of Harary's Bipartition Theorem by proving (i) $\Rightarrow$ (ii).

Proof. Without loss of generality, assume Σ is connected. Pick a spanning tree T and a root vertex $r \in V(T)$. For each vertex $v \in V$, there exists a unique path T_{rv} from r to v in T. Define $\zeta(v) = \sigma(T_{rv})$. Switch to Σ^ζ. Now all edges in T become $+$. If $e \notin T$ then there exists a unique $C_e \subseteq T \cup e$. Then $+ = \sigma(C_e) = \sigma^\zeta(e)\sigma^\zeta(C_e \setminus e) = \sigma^\zeta(e)$. Therefore, $\sigma^\zeta(e) = +$ and Σ^ζ is all $+$.

Let $V_1 = \zeta^{-1}(+)$ and $V_2 = \zeta^{-1}(-)$. We claim this is a Harary Bipartition, so we are done. □

We will begin next time by proving that this is indeed a Harary Bipartition.

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[3]Named by me in honor of Harary.

Notes for 23 Jan. 2017 – Chris Eppolito.

We wish to finish the proof of the following theorem:

Theorem 6 (Harary's Balance Theorem [6])**.** *For a signed graph Σ without half edges, the following are equivalent:*

(o) *Σ is balanced.*
(i) *Every circle of Σ is positive.*
(ii) *There is a Harary bipartition of Σ.*
(iii) *For each pair u, v of vertices, every uv-path has the same sign as every other.*

All that remains from the previous lecture is a proof that (i) $\Longrightarrow$ (ii).

Proof that (i) $\Longrightarrow$ (ii). We may assume Σ has connected underlying graph $|\Sigma| = (V, E)$. Let T be a spanning tree of $|\Sigma|$ with root vertex r. For each vertex v of Σ, let T_{rv} denote the path in T connecting r to v. By Lemma 1 there is a switching function $\zeta : V \to \{+, -\}$ such that Σ^ζ is positive on T. Now let $e \in E \setminus T$ and consider the fundamental circle $C_e \subseteq T \cup \{e\}$ of e with respect to T. By assumption (i),

$$\begin{aligned} + = \sigma(C_e) = \sigma^\zeta(C_e) &= \prod_{a \in C_e} \sigma^\zeta(a) \\ &= \Big(\prod_{t \in T \cap C_e} \sigma^\zeta(t) \Big) \Big(\prod_{a \in C_e \setminus T} \sigma^\zeta(a) \Big) \\ &= \Big(\prod_{t \in T \cap C_e} + \Big) \sigma^\zeta(e) = \sigma^\zeta(e). \end{aligned}$$

In particular, Σ^ζ has all positive edges. Now consider the bipartition $V = \zeta^{-1}(+) \sqcup \zeta^{-1}(-)$. Suppose $e{:}uv$ has $u \neq v$, and note that $+ = \sigma^\zeta(e) = \zeta(u)\sigma(e)\zeta(v)$ gives $\sigma(e) = \zeta(u)\zeta(v)$. If $\sigma(e) = +$, then this identity gives $\zeta(u) = \zeta(v)$; in particular u and v belong to the same part of our bipartition. If $\sigma(e) = -$, then the identity gives $\zeta(u) = -\zeta(v)$; thus u and v belong to different parts of our bipartition. Hence $\zeta^{-1}(+) \sqcup \zeta^{-1}(-)$ is a Harary bipartition of V. $\square$

2. Characterization

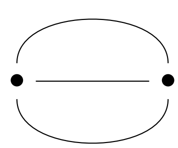
What sets of circles can be the negative circle class of a signed graph? The answer is simple. The proof is not so simple, mainly because I want to show how to use linear algebra to prove an interesting theorem of signed graph theory.

We fix the following notations.

Definition 9. Let Γ be a graph and let $\Sigma = (|\Sigma|, \sigma)$ be a signed graph. Then

$$\begin{aligned} \mathscr{C}(\Gamma) &:= \{C \leq \Gamma \mid C \text{ is a circle in } \Gamma\}, \\ \mathscr{B}(\Sigma) &:= \{C \in \mathscr{C}(|\Sigma|) \mid \sigma(C) = +\}. \end{aligned}$$

Our next theorem is closely tied to the aptly named theta graphs. A graph is a *theta graph* when it is a subdivision of the graph depicted below (three links with the same endpoints):



We often express a theta graph as the union of two of its three distinct circles.

Lemma 7. *Every signed theta graph has an even number of negative circles.*

Proof. Every theta graph Γ can be expressed as a union of three distinct paths sharing only endpoints. In particular $\Gamma = P_1 \cup P_2 \cup P_3$, and there are precisely three distinct circles of Γ, namely $C_{\pi(1)} = P_{\pi(2)} \cup P_{\pi(3)}$ for cyclic permutations π of $\{1, 2, 3\}$; moreover, $\sigma(C_{\pi(1)}) = \sigma(P_{\pi(2)})\sigma(P_{\pi(3)})$, and one easily checks that for all triples $t \in \{+, -\}^3$ the pairwise products of the entries of t must yield evenly many $-$'s. The result follows. $\square$

For a theta graph $\Gamma = C_1 \cup C_2 \cup C_3$ we have $C_1 \oplus C_2 \oplus C_3 = \varnothing$, where $\oplus$ denotes the symmetric difference of sets. This is proved by decomposing Γ into its three paths as in the preceding proof.

Definition 10. Let Γ be a graph. A subset $\mathscr{L} \subseteq \mathscr{C}(\Gamma)$ is a *linear class* when for any $C, D \in \mathscr{L}$, if $C \cup D$ is a theta graph then $C \oplus D \in \mathscr{L}$.

The following theorem concerning linear classes of circles is quite powerful.

Theorem 8 (Tutte's Path Theorem)**.** *Let Γ be a connected, inseparable graph with a linear class of circles $\mathscr{L} \subseteq \mathscr{C}(\Gamma)$. For all $A, B \in \mathscr{C}(\Gamma)$ there is a sequence*

$$A = C_0, C_1, C_2, \ldots, C_l = B$$

such that $C_i \notin \mathscr{L}$ for all $i \in [l]$ and $C_{i-1} \cup C_i$ is a theta graph in Γ for all $i \in [l]$.

Proof. Omitted. [**TZ: Give a reference.**] $\square$

This theorem admits a nice interpretation when viewed in the following manner. Let Γ be a graph and define a new graph $\Theta = \Theta(\Gamma)$ by defining its vertices and edges as follows:

$$\begin{aligned} V(\Theta) &= \mathscr{C}(\Gamma), \\ E(\Theta) &= \{\{C, D\} \mid C \cup D \text{ is a theta graph in } \Gamma\}. \end{aligned}$$

The theorem asserts that if Γ is connected and inseparable,[4] then for every linear class of circles $\mathscr{L} \subseteq \mathscr{C}(\Gamma)$ in Γ and all $A, B \in V(\Theta)$ there is an AB-path in Θ avoiding $\mathscr{L}$ internally.

Definition 11. Let Γ be a graph. A subset $\mathscr{B} \subseteq \mathscr{C}(\Gamma)$ is *theta additive* in Γ when in every theta subgraph $C_1 \cup C_2 \cup C_3$ of Γ, evenly many of C_1, C_2, C_3 are in $\mathscr{B}^c$.

We shall use the above theorem to obtain the following result on signed graphs.

Theorem 9. *Let Γ be a graph with $\mathscr{B} \subseteq \mathscr{C}(\Gamma)$. There is a signature σ of Γ for which $\mathscr{B}(\Gamma, \sigma) = \mathscr{B}$ precisely when $\mathscr{B}$ is theta additive.*

The proof will rest on some algebra over the field $\mathbb{F}_2$. We make the following observations.

Definition 12. For a set E with $S \subseteq E$, the characteristic function of S in E is denoted by $\mathbf{1}_S$.

Lemma 10. *Let E be a set. The power set $\mathscr{P}(E)$ with symmetric difference as addition and the obvious $\mathbb{F}_2$-action forms a vector space isomorphic to the function space $\mathbb{F}_2^E$.*

Proof. The desired correspondence is $S \leftrightarrow \mathbf{1}_S$. What remains is a straightforward check. $\square$

[4]This is equivalent to the statement that the cycle matroid of Γ is connected.

Let Γ be a graph. Given a signing $\sigma : E(\Gamma) \to \{+,-\}$ of Γ, one extends σ to a mapping

$$\sigma : \mathscr{C}(\Gamma) \to \{+,-\} : C \mapsto \prod_{e \in C} \sigma(e)$$

It then follows that the mapping given below will be important:

$$\mathbf{1}_{\mathscr{B}^c(\sigma)} : \mathbb{F}_2^{\mathscr{C}(\Gamma)} \to \mathbb{F}_2 : C \mapsto \begin{cases} 1 & \text{if } \sigma(C) = -, \\ 0 & \text{otherwise.} \end{cases}$$

Notes for 25 Jan. 2017 – Micah Loverro.

Given a set of circles $\mathscr{B} \subseteq \mathscr{C}(\Gamma)$, we are proving:

1. If there is a signature σ such that $\mathscr{B} = \mathscr{B}(\sigma)$, then $\mathscr{B}$ is *theta additive*, i.e., every theta subgraph has an even number of circles in $\mathscr{B}$.
2. If $\mathscr{B}$ is theta additive, then there is σ such that $\mathscr{B} = \mathscr{B}(\sigma)$.

Part 1 is easy, and we were in the process of proving 2.

We have $\mathbf{1}_{\mathscr{B}} \colon \mathscr{C}(\Gamma) \to \mathbb{F}_2$.

The *binary cycle space* is $Z := Z_1(\Gamma; \mathbb{F}_2) := \langle \mathscr{C}(\Gamma) \rangle \leq \mathbb{F}_2^E$.

If $\mathbf{1}_{\mathscr{B}}$ extends to a homomorphism $\overline{\sigma} : Z \to \mathbb{F}_2$ then by basic linear algebra, $\overline{\sigma}$ extends to a homomorphism of the larger vector space $(\mathscr{P}(E), \oplus) \cong \mathbb{F}_2^E \to \mathbb{F}_2$.

The restriction of such a $\overline{\sigma}$ to E gives a sign function $\sigma \colon E \to \{+,-\}$ under the obvious identification $\mathbb{F}_2 \cong \{+,-\}$ as groups. Then it follows that $\mathscr{B}(\Gamma; \sigma) = \mathscr{B}$.

Let us now attempt to extend $\mathbf{1}_{\mathscr{B}}$ to $\overline{\sigma} : Z \to \mathbb{F}_2$ such that $\overline{\sigma}$ is a well-defined homomorphism.

In fact, well-definedness alone will imply we have a homomorphism. To see this, suppose $A = \sum_{i=1}^k A_i$ and $B = \sum_{j=1}^l B_j$ are sums of circles. Then $A \oplus B = \sum_{i=1}^k A_i \oplus \sum_{j=1}^l B_j$ is a sum of circles. So, $\overline{\sigma}(A \oplus B)$ is defined as $\sum_{i=1}^k \overline{\sigma}(A_i) + \sum_{j=1}^l \overline{\sigma}(B_j) = \overline{\sigma}(A) + \overline{\sigma}(B)$.

To see $\overline{\sigma}$ is well-defined, suppose $A_1 \oplus \cdots \oplus A_k = B_1 \oplus \cdots \oplus B_l$. Then we want to show $\overline{\sigma}(A_1 \oplus \cdots \oplus A_k) = \overline{\sigma}(B_1 \oplus \cdots \oplus B_l)$. Equivalently, we want to show that whenever $C_1 \oplus \cdots \oplus C_m = \varnothing$ in Z, we have $\sum_{i=1}^m \overline{\sigma}(C_i) = 0$.

(This proof is incomplete and will be fixed in the next day's notes).

Suppose we have a chain of circles $C_0, C_1, \ldots, C_r$ such that every $C_{i-1} \cup C_i$ is a theta graph. Then $D_i := C_{i-1} \oplus C_i$ is a circle. Thus $\mathbf{1}_{\mathscr{B}}(C_{i-1}) + \mathbf{1}_{\mathscr{B}}(C_i) = \mathbf{1}_{\mathscr{B}}(D_i)$.

We define a *linear class of circles* as a set $\mathscr{B} \subseteq \mathscr{C}(\Gamma)$ that is theta additive.

We will make uses of the following theorem due to Tutte:

Theorem 11. *Let Γ be inseparable. If $\mathscr{B}$ is a linear class of circles, and $C, C' \in \mathscr{B}$ then there is a path of circles $C = C_0, \ldots, C_r = C'$ such that each $C_1, \ldots, C_{r-1}$ is not in $\mathscr{B}$ and $C_{i-1} \cup C_i$ is a theta graph for all $i = 1, \ldots, r$.*

Notes for 27 Jan. 2017 – Amelia Mattern.

Definition 13. Given a graph Γ. A class $\mathscr{B}$ of circles is *additive* if, whenever $C, C_1 \ldots, C_k \in \mathscr{C}$, such that $C = C_1 \oplus \ldots \oplus C_k$, then:

If an even number of $C_1, \ldots, C_k$ are not elements of $\mathscr{B}$, then $C \in \mathscr{B}$.
If an odd number of $C_1, \ldots, C_k$ are not elements of $\mathscr{B}$, then $C \notin \mathscr{B}$.

Equivalently, $\mathscr{B}$ is additive if $\mathscr{B} = C \cap \mathscr{S}$, where $\mathscr{S}$ is a linear subspace of $\langle \mathscr{C} \rangle$ of codimension at most 1.

Also equivalently, if $C = C_{k+1}$ and $C_1 \oplus \ldots C_{k+1} = \varnothing$, then $\mathscr{B}$ is additive if an even number of C_i's are not in $\mathscr{B}$.

An Explanation of $\mathscr{S}$:

First note that $\mathscr{S} = \langle \mathscr{B} \rangle$. So we have $\langle \mathscr{C} \rangle = \langle \mathscr{B} \rangle \cup \langle \mathscr{B} + C_0 \rangle$ where C_0 is any element of $\langle \mathscr{C} \rangle \setminus \langle \mathscr{B} \rangle$. Therefore, if $C_1, \ldots, C_m \notin \langle \mathscr{C} \rangle \setminus \langle \mathscr{B} \rangle$, then $C_1 \oplus \ldots \oplus C_m = (C_1 \oplus C_0) \oplus \ldots \oplus (C_m \oplus C_0) +$ (sum of m C_0's). Note that (sum of m C_0's) $= \varnothing$ if m is even and $= C_0$ if m is odd. So $C_1 \oplus \ldots \oplus C_m \in \langle \mathscr{B} \rangle$ if m is even, and $C_1 \oplus \ldots \oplus C_m \notin \langle \mathscr{B} \rangle$ if m is odd.

Lemma 12. *Given a signed graph $\Sigma = (\Gamma, \sigma), \mathscr{B} \subseteq \mathscr{C}$ is additive if and only if it satisfies the theta graph condition.*

Note that $C = C_1 \oplus C_2$ does not imply that $C \cup C_1 \cup C_2$ is a theta graph.

Proof. ($\Rightarrow$) Suppose $\mathscr{B}$ is additive. Then for any theta subgraph of Γ with circles C_1, C_2, C_3, we have $C_1 \oplus C_2 \oplus C_3 = \varnothing$. Thus by additivity, an even number of C_i's are not in $\mathscr{B}$. Thus $\mathscr{B}$ satisfies the theta graph condition.

($\Leftarrow$) Assume Γ is inseparable, $C_1 \oplus \ldots \oplus C_n = \varnothing$, and an odd number of C_i are not in $\mathscr{B}$. By choice of notation assume $C_1 \notin \mathscr{B}$. Assume Γ has the fewest possible edges for the previous properties to occur. Let $e \in C_1$, then e is in at least two C_i's. By choice of notation let $e \in C_1 \cap C_2$. By Tutte's path theorem C_1 and C_2 are joined by a "path" of circles, $C_1 = C'_0, C'_1, \ldots, C'_l = C_2$ where $C'_1, \ldots, C'_{l-1} \notin \mathscr{B}$, $C'_{i-1} \cup C'_i$ is a theta graph, and $e \notin C'_i$ for $0 < i < l$. Let $D_i = C'_i \oplus C'_{i-1}$. Then $e \notin D_i$ for $1 < i < l$ and $e \in D_1 \cap D_l$. Also, $D_2 \oplus \ldots \oplus D_{l-1} = C'_1 \oplus C'_{l-1} \subseteq \Gamma \setminus \{e\}$. Therefore, an even number of D_i's and C'_1, C'_{l-1} are in $\mathscr{B}$. Also,

$$C_1 \oplus C_2 = C'_0 \oplus C'_l = D_1 \oplus D_l \oplus (D_2 \oplus \ldots \oplus D_{l-1}).$$

This implies $C_1 \oplus C_2 = (D_1 \oplus C'_1) \oplus (D_l \oplus C'_{l-1})$, which means $(C_1 \oplus D_1 \oplus C'_1) \oplus (C_2 \oplus D_l \oplus C_{l-1}) = \varnothing$. By theta additivity, $\bar{\sigma}(C_1) + \bar{\sigma}(D_1) + \bar{\sigma}(C'_1) = 0$ and $\bar{\sigma}(C_2) + \bar{\sigma}(D_l) + \bar{\sigma}(C'_{l-1}) = 0$. By additivity in $\Gamma \setminus \{e\}$,

$$\bar{\sigma}(C'_1) + \bar{\sigma}(C'_{l-1}) + \sum_{i=2}^{l-1} \bar{\sigma}(D_i) = 0.$$

So $\bar{\sigma}(C_1) + \bar{\sigma}(C_2) = \sum_{i=1}^{l} \bar{\sigma}(D_i) = 0$ since $D_i \in \mathscr{B}$. Then since $\bar{\sigma}(C_1) = 1, \bar{\sigma}(C_2) = 1$. Thus $C_2 \notin \mathscr{B}$.

Suppose $C_1, \ldots, C_{2k}$ contain e and $C_{2k+1}, \ldots, C_l$ do not contain e. We showed $C_1 \notin \mathscr{B}$ implies $C_2 \notin \mathscr{B}$ and $C_1 \oplus C_2 = D_1 \oplus \ldots D_l$ with $e \notin D_i$ and $D_i \in \mathscr{B}$. Similarly, if $k > 1, C_3 \oplus C_4 = D_1^{34} \oplus \ldots \oplus D_{l_{34}}^{34}$.

Note: This proof is incomplete, and has a few things incorrect. The correct/completed proof will be done next time. $\square$

Notes for 30 Jan. 2017 – Ted Ofner.

We were showing that if a class of circles $\mathscr{B}$ is theta-additive, then $\mathscr{B}$ is additive. Additivity of $\mathscr{B}$ can be restated as: if $C_i \in \mathscr{B}$ for $i = 1, \dots, n$ and $C_1 \oplus C_2 \oplus \cdots \oplus C_n = \varnothing$, then $\sum_{i=1}^{n} \bar{\sigma}(C_i) = 0$. Here $\bar{\sigma}$ is the linear extension to the whole cycle space of the function of circles given by

$$\bar{\sigma}(C) = \begin{cases} 0 & \text{if } C \in \mathscr{B}, \\ 1 & \text{if } C \notin \mathscr{B}. \end{cases}$$

This latter function, with domain $\mathscr{C}$, is known as the *indicator function* or *characteristic function* of $\mathscr{B}$ as a subset of $\mathscr{C}$; it is usually written $1_{\mathscr{B}}$.

If the C_i are pairwise edge disjoint, linear extension directly implies that $\sum_{i=1}^{n} \bar{\sigma}(C_i) = \bar{\sigma}(\bigoplus_{i=1}^{n} C_i) = 0$.

If the C_i are not pairwise edge disjoint, two of the C_i intersect. By choice of notation, say $C_1 \cap C_2 \neq \varnothing$. Let $e \in C_1 \cap C_2$. Assume by way of contradiction that the C_i form a counterexample to our proposition. This counterexample contains e as an edge. Therefore, there is a counterexample collection of circles containing e such that the number of C_i containing e is minimal. Without loss of generality, we assume our C_i form this minimal counterexample.

Since C_1 and C_2 share an edge, they are contained in the same block of Σ. This allows us to apply the Tutte path theorem:

Theorem 13 (Tutte's Path Theorem)**.** *Let $\mathscr{L}$ be a theta-additive class of circles in a graph Γ, and let C, C' be circles in Γ contained in a single block. There exists a sequence of circles $C = C_0, C_1, \dots, C_n = C'$ such that*

$$\begin{gathered} C_1, \dots, C_{n-1} \notin \mathscr{L} \text{ and} \\ C_{i-1} \cup C_i \text{ is a theta graph for } i = 1, \dots, n. \end{gathered}$$

We apply this theorem to C_1 and C_2 using the theta-additive class of circles C such that $e \notin C$. This gives us a sequence of circles $C_1 = C'_0, C'_1, \dots, C'_l = C_2$ with

$$e \in C'_0, \dots, C'_l \text{ and } C'_{i-1} \cup C'_i \text{ is a theta graph for } i = 1, \dots, l.$$

Let $D_i = C'_{i-1} \oplus C'_i$ for $i = 1, \dots, l$. Since $C'_{i-1} \cup C'_i$ is a theta graph, each D_i is a circle. Since $e \in C'_0, \dots, C'_l$, $e \notin D_i$ for all $i = 1, \dots, l$. Moreover,

$$\begin{aligned} D_1 \oplus D_2 \oplus \cdots \oplus D_n &= (C'_0 \oplus C'_1) \oplus (C'_1 \oplus C'_2) \oplus \cdots \oplus (C'_{l-1} \oplus C'_l) \\ &= C'_0 \oplus C_l = C_1 \oplus C_2. \end{aligned}$$

Also, $\bar{\sigma}(D_i) = \bar{\sigma}(C'_{i-1}) + \bar{\sigma}(C'_i)$ from our assumption that $\mathscr{B}$ is theta-additive. Therefore,

$$\begin{aligned} \sum_{i=1}^{l} \bar{\sigma}(D_i) &= \sum_{i=1}^{l} (\bar{\sigma}(C'_{i-1}) + \bar{\sigma}(C'_i)) \\ &= \bar{\sigma}(C'_0) + \bar{\sigma}(C'_l) = \bar{\sigma}(C_1) + \bar{\sigma}(C_2). \end{aligned}$$

Now consider the class of circles $\{D_1, \ldots, D_l, C_3, C_4, \ldots, C_n\}$. Since none of the D_i contains e, this class of circles has fewer instances of e as an edge than $\{C_1, \ldots, C_n\}$. By minimality of our counterexample, this class must be additive. We now have

$$\begin{aligned}
0 &= \bar{\sigma}(C_1 \oplus \cdots \oplus C_n) \\
&= \bar{\sigma}(D_1 \oplus D_2 \oplus \cdots \oplus D_l \oplus C_3 \oplus \cdots \oplus C_n) \\
&= \bar{\sigma}(D_1) + \bar{\sigma}(D_2) + \cdots + \bar{\sigma}(D_l) + \bar{\sigma}(C_3) + \cdots + \bar{\sigma}(C_n) \\
&= \bar{\sigma}(C_1) + \cdots + \bar{\sigma}(C_n).
\end{aligned}$$

Thus $\mathscr{B}$ is additive. This allows us to finish our proof of the following theorem.

Theorem 14. *$\mathscr{B}$ is a theta-additive class of circles $\iff$ there exists a signature σ such that $\mathscr{B}$ is the collection of positive circles of σ.*

Proof. ($\Leftarrow$) has been covered previously and is a simple calculation.

For ($\Rightarrow$), choose a maximal forest in Γ, fix a basepoint for each component, and switch so that the chosen forest is all positive. If e is an edge not in the forest, let C_e be the unique circle through the basepoint that has e as its only non-forest edge. Define the sign of e to be $\bar{\sigma}(C_e)$ where $\bar{\sigma}$ is the linear extension of the indicator function of $\mathscr{B}$. It is a simple calculation that this returns $\mathscr{B}$ as its set of positive circles. $\square$

Notes for 1 Feb. 2017 – Josh Carey.

3. No Negative or No Positive Circles

We want to know under what circumstances a signed graph will have no negative circles. Since half edges are irrelevant, we can assume there are none. There is exactly one way to have no negative circles: Σ must be balanced; equivalently $\Sigma \sim (|\Sigma|, +)$.

How can we determine whether or not Σ does have a negative circle? To prove a positive or negative answer is easy, in a sense:

(i) Produce a Harary bipartition; this implies Σ is balanced.
(ii) Find a negative circle; this implies Σ is unbalanced.

The bigger question is how to find the bipartition or the negative circle. We can proceed algorithmically in at least two ways (which are quite similar). First, check a limited number of circle signs:

F1. Find a spanning tree T.
F2. Test all the fundamental circles $C_T(e)$ of edges not in T. [**FUND. CIRCLES DEFINED?**] If one is negative, Σ is unbalanced. If none is, Σ is balanced.

This method doesn't immediately produce a Harary bipartition. But the second method does: we apply switching:

S1. Find a spanning tree T.
S2. Switch Σ so T is all positive.
S3. Check the sign of every non-tree edge. If one is negative, Σ is unbalanced. If all are positive, Σ is balanced.

The sign of e in the second method is the sign of its fundamental circle in the first method, so the two are closely related. In the switching method the Harary bipartition of a balanced graph appears clearly as $\{\zeta^{-1}(+), \zeta^{-1}(-)\}$, where ζ is the switching function used in the second step. Both methods are fast, as they involve fewer than $\#E$ steps.

Now let's reverse the first question. Under what circumstances does a signed graph have no positive circles? A theta graph has 1 or 3 positive circles, so we want no theta subgraphs. If a block has more than one circle, it is a theta subgraph. So every block of Σ is an isolated vertex, an isthmus, or a negative circle. I call a graph of this kind a *cactus forest* (a cactus is a connected graph in which every block is a circle); when it is signed so every circle is negative it is a *contrabalanced cactus forest* (*contrabalance* meaning there are no balanced circles).

The great difference between the two answers is a clear proof that $+$ and $-$ are very different.[5]

4. Frustration–Deletion–Negation

4.1. Physics!

[ONLY SOME IS PHYSICS – SORT THIS OUT.]

I want to start this topic with a connection to physics. A *state* of a signed graph Σ is a mapping $s : V \to \{+, -\}$. That is, a state of Σ is an assignment of a sign to each vertex. For an edge $e{:}vw$, if $s(v)s(w) = \sigma(e)$, then e is *satisfied*. If $s(v)s(w) \neq \sigma(e)$, then e is *frustrated*. A signed graph (with no half or loose edges; they don't use them) is therefore called (in physics) *frustrated* if it is unbalanced and *satisfied* if it is balanced (but I won't use physics terminology except for the excellent names "frustration index" and "number"—my own coinages, approved by Harary!).

Signed graphs appear in the non-ferromagnetic Ising model of statistical physics. In this model we have a graph with weighted edges, the weights being real numbers, and we assign a state to the graph. The values on the vertices are called "spins" and are assumed to be either up $(+)$ or down $(-)$. **[MORE]**

A *balancing edge set* is a set $S \subseteq E$ such that negating the signs on S makes Σ balanced. The *frustration index* $l(\Sigma) := \min\{\#S : S \text{ is a balancing set of edges}\}$. If we use s to switch, we get Σ^s where the set of unsatisfied edges in the all-positive state equals the set of unsatisfied edges of Σ in state s. The former is $E^-(\Sigma^s)$, which is a balancing set.

[ADD ENERGY = FRUST.]

Notes for 3 Feb. 2017 – Chris Eppolito.

4.2. Negation and Deletion Sets of Edges.

First we recall several definitions:

Definition 14. Let Σ be a signed graph. A set $S \subseteq E(\Sigma)$ is a *negation set* in Σ when the signed graph Σ' obtained from Σ by negating all edges in S is balanced. A set $S \subseteq E(\Sigma)$ is a *deletion set* in Σ when the signed graph $\Sigma \setminus S$ is balanced.

[5] In case you wondered, this is not trivial. For "signed graphs" in knot theory there is no difference between the two signs; one can reverse them and nothing changes. I consider these signs to be colors and call the graphs *sign-colored graphs*.

Definition 15. A *cut-set* in a graph Γ is a set $C \subseteq E(\Gamma)$ such that $\Gamma \setminus (E(\Gamma) \setminus C)$ is bipartite.

Proposition 15. *Let Γ be a graph. The minimal deletion sets of $-\Gamma$ are precisely the complements of maximal cut-sets in Γ.*

The Harary Balance Theorem implies that $-\Gamma$ is balanced precisely when Γ is bipartite. This suggests that balance may be a signed-graph generalization of bipartiteness.

Proof. Any cut-set S of Γ determines a bipartite subgraph $(V(\Gamma), S) \leq \Gamma$, so S^c is necessarily a deletion set of $-\Gamma$; furthermore, maximality of S as a cutset implies minimality of S^c as a deletion set. Likewise any deletion set S of $-\Gamma$ has the property that $-\Gamma : S^c$ is balanced giving that $(V(\Gamma), S^c)$ is bipartite; furthermore minimality of S as a deletion set yields maximality of S^c as a cut-set. Hence the desired result follows. ☐

Proposition 16. *Let Σ be a signed graph.*

(1) Every negation set of Σ is a deletion set.
(2) Every minimal deletion set of Σ is a minimal negation set.[6]

Lemma 17. *A set S is a deletion set for Σ if and only if S intersects every element of $\mathscr{B}^c(\Sigma)$.*

In other words, deletion sets of Σ are precisely transversals of $\mathscr{B}^c(\Sigma)$.

Proof. Let Σ be a signed graph. Notice immediately that we have the following equality:

$$\mathscr{B}^c(\Sigma \setminus S) = \{C \in \mathscr{B}^c(\Sigma) \mid C \cap S = \varnothing\}$$

Furthermore $S \subseteq E(\Sigma)$ is a deletion set for Σ precisely when $\Sigma \setminus S$ is balanced under the induced signing from Σ. Hence we complete the proof as follows:

$$\begin{aligned} S \subseteq E(\Sigma) \text{ is a deletion set for } \Sigma &\iff \Sigma \setminus S \text{ is balanced} \\ &\iff \mathscr{B}^c(\Sigma \setminus S) = \varnothing \\ &\iff \text{for all } C \in \mathscr{B}^c(\Sigma) \text{ we have } C \cap S \neq \varnothing \\ &\iff \text{every element of } \mathscr{B}^c(\Sigma) \text{ intersects } S. \end{aligned}$$ ☐

Lemma 18. *The class of deletion sets of Σ is invariant under switching.*

Proof. A result from before gives $\mathscr{B}(\Sigma) = \mathscr{B}(\Sigma^\zeta)$, and what remains follows by Lemma 17. ☐

Lemma 19. *The class of negation sets of Σ is invariant under switching.*

Denote the signed graph obtained by negating signs on edges in S by $\mathrm{Neg}(\Sigma, S) = (|\Sigma|, \sigma_S)$.

Proof. If S is a negation set of Σ, then $\mathrm{Neg}(\Sigma, S)$ is balanced. Thus there is a switching ζ with $\mathrm{Neg}(\Sigma, S)^\zeta$ all positive. Next one notes that $(\sigma_S)^\zeta = (\sigma^\zeta)_S$, which in turn implies that $\mathrm{Neg}(\Sigma, S)^\zeta = \mathrm{Neg}(\Sigma^\zeta, S)$. In particular $\mathrm{Neg}(\Sigma, S)$ is balanced precisely when $\mathrm{Neg}(\Sigma^\zeta, S)$ is balanced, by switching invariance of balance. Hence S is a negation set of Σ, as desired. ☐

Proof of Proposition 16. We may assume without loss of generality that $|\Sigma|$ has only loops and links and is connected.

[6] "Minimal" means "minimal with respect to containment."

Part 1: Suppose $S \subseteq E(\Sigma)$ is a negation set for Σ, and consider the graph $\Sigma \setminus S$; notice that $\Sigma \setminus S \subseteq \mathrm{Neg}(\Sigma, S)$ and $\Sigma \setminus S \subseteq \Sigma$ trivially. Moreover, every (signed) circle C in $\Sigma \setminus S$ is positive in $\mathrm{Neg}(\Sigma, S)$ as S is a negation set. Hence S is a deletion set as desired.

Part 2: Suppose $S \subseteq E(\Sigma)$ is a minimal deletion set for Σ, and let C be an arbitrary circle in $\mathrm{Neg}(\Sigma, S)$. By minimality of S and connectedness of $|\Sigma|$ there is a spanning tree T of Σ contained in $\Sigma \setminus S$. Now switch Σ by a function $\zeta = \zeta_T$ such that $\sigma^\zeta(t) = +$ for all $t \in T$. By considering fundamental circuits in $\Sigma \setminus S$ we see that $\Sigma^\zeta \setminus S$ is all positive. Hence $E_-(\Sigma^\zeta) \subseteq S$ is a deletion set of Σ, so by minimality of S, $S = E_-(\Sigma^\zeta)$. What remains follows by invariance of the negation sets of Σ under switching. To see that S is a minimal negation set we need only note that if $S' \subseteq S$ is a negation set, then it is a deletion set by Part 1, so $S' = S$ by minimality of S.

We conclude that the original statement is true. □

Proposition 20. *The minimal negation sets of* Σ *are precisely the minimal deletion sets.*

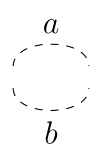
Proof. By Part (2) of Proposition 16, the minimal deletion sets of Σ form a subclass of the minimal negation sets of Σ.

Conversely, let $S \subseteq E(\Sigma)$ be a minimal negation set of Σ. Now S is a deletion set of Σ by Part (1) of Proposition 16, so there is a minimal deletion set $S' \subseteq S$ by finiteness of $E(\Sigma)$; finally, by Part (2) of Proposition 16 S' is a minimal negation set of Σ, and minimality of S gives $S = S'$. Thus the minimal negation sets of Σ form a subclass of the minimal deletion sets of Σ. Hence the result holds. □

Example 1. Let Σ be a signed graph.

(1) Deletion sets need not be negation sets.
(2) Negation sets need not be minimal deletion sets.

Proof. Consider the all-negative circle depicted below:



(1): Clearly $\{a\}$ is a deletion set, but not a negation set.
(2): Clearly $\{a, b\}$ is a negation set, but the (unique) minimal deletion set is $\varnothing$.
This example shows that Proposition 16 is tight. □

4.3. **Results on the Frustration Index.**

Definition 16. Let Σ be a signed graph. The *frustration index* is

$$l(\Sigma) := \min \{\#S \mid S \subseteq E(\Sigma) \text{ is a deletion set of } \Sigma\}.$$

The *frustration number* is

$$l_0(\Sigma) := \min \{\#X \mid X \subseteq V(\Sigma) \text{ and } \Sigma \setminus X \text{ is balanced}\}.$$

The following proposition gives a nice interpretation of these concepts in terms of the structure of the underlying graph in a special case.

Proposition 21. *Let Γ be a graph. Then*

$$l(-\Gamma) = \min\{\#S \mid S \subseteq E(\Gamma) \text{ and } \Gamma \setminus S \text{ is bipartite}\},$$
$$l_0(-\Gamma) = \min\{\#X \mid X \subseteq V(\Gamma) \text{ and } \Gamma{:}X \text{ is bipartite}\}.$$

Proof. By the Harary Balance Theorem it follows that $-\Gamma$ is balanced precisely when Γ is bipartite. What remains follows by examining the appropriate definitions. □

Proposition 22. *For every signed graph Σ without half edges,*

$$l(\Sigma) = \min\left\{\#E^{-}(\Sigma^X) \mid X \subseteq V(\Sigma)\right\}$$

(If there are half edges, they have to be added into the count to get $l(\Sigma)$.)

Proof. Left as an exercise. □

Definition 17. The *maximum frustration index* of a graph Γ is

$$l_{\max}(\Gamma) := \max\{l(\Gamma, \sigma) \mid \sigma \text{ is a signing of } \Gamma\}.$$

Example 2. The graph $K_{4,4}$ has maximum frustration index 4.

The following (open) problem has been solved for $r \leq 5$:

Problem 1 (Open)**.** Compute $l_{\max}(K_{r,s})$ for all $r, s \in \mathbb{Z}_{>0}$.

The following simple result was proved by Petersdorf.

Proposition 23 (?)**.** *For $n \in \mathbb{Z}_{>0}$, $l_{\max}(K_n) = l(-K_n) = \binom{n-2}{2}$.*

Exercise: Is the correct value of $l(-K_n)$ actually $\binom{n-2}{2}$? If not, correct it!

Proof. Another exercise. □

Petersdorf also proved something that is less simple: $[-K_n]$ is uniquely maximizing, i.e., it is the only switching class that maximizes the frustration index of a signed complete graph. This raises the following natural question:

Problem 2 (Open)**.** Characterize all graphs Γ for which $l_{\max}(\Gamma) = l(-\Gamma)$.

One might also consider the following question.

Definition 18. A graph (or signed graph) is said to *decompose* into graphs of the class $\mathfrak{C}$ when it can be expressed as an edge-disjoint union of graphs in the class $\mathfrak{C}$.

Problem 3 (Open)**.** Which signed graphs decompose into positive, or negative, circles?

One can obtain obvious necessary conditions, but sufficient conditions are much harder.

Notes for 6 Feb. 2017 – Micah Loverro.

5. Working with Signed Graphs

Let's discuss some basic operations on signed graphs.

Disjoint union is the usual disjoint union of the underlying graphs with the obvious sign function.

A *subgraph* (or sub-signed graph, or signed subgraph) of a signed graph (Γ, σ) is obtained by taking a subgraph Γ' of Γ and letting σ' be the restriction of σ to $E^*(\Gamma')$. A subgraph always inherits the signs of Σ.

Union If two signed graphs Σ_1 and Σ_2 share a common signed subgraph Σ, their union along Σ can be thought of as gluing them together along Σ. In contrast to unsigned graph theory, in order for this to make sense σ_1 and σ_2 must have the same values when restricted to $E^*(\Sigma)$.

The *positive subgraph and negative subgraph* of Σ are defined to be $\Sigma^+ := (E^+(\Sigma), V(\Sigma))$ and $\Sigma^- := (E^-(\Sigma), V(\Sigma))$, respectively. They are unsigned graphs.

Deleting a vertex v means, as usual, removing v from the vertex set, and removing all edges e incident to v from the edge set.

Reducing a vertex means we remove v from the vertex set, but we do not delete any edges. Instead, we may be left with half edges or loose edges.

Deleting an edge is simple enough. Just remove e from the edge set. The new sign function is σ restricted to $E \setminus \{e\}$.

Induced Subgraph For a subset of vertices $W \subseteq V$, the subgraph induced by W is denoted by $\Sigma\colon W$. The vertex set is W and the edge set is $E\colon W = \{e \in E \mid v(e) \subseteq W \text{ and } v(e) \neq \varnothing\}$.

Let's write $q(\Gamma)$ (a random letter) for the number of switching isomorphism classes of signatures of the graph Γ. For example $q(K_{4,4}) = 10$.

We define $b(\Sigma)$ as the number of balanced (connected) components of Σ, and $c(\Sigma)$ as the total number of components.

We have the notion of a balancing edge or edge set or vertex or vertex set X. X is a *total balancing* (edge, edge set, vertex, or vertex set) if $\Sigma \setminus X$ is balanced. X is a *partial balancing* (edge, edge set, vertex, or vertex set) if $b(\Sigma \setminus X) > b(\Sigma)$.

If Σ is connected, one way we can compute a minimal edge balancing set is by switching so that some spanning tree is all positive. Then the negative edges comprise a minimal (total) edge balancing set. If Σ has more than one connected component, this process can be applied to each individual component.

There are two partitions associated with a signed graph. Just as for an unsigned graph we have $\pi(\Sigma)$, the partition of the vertex set V into vertex sets of connected components. Unique to signed graphs is the *balanced partial partition*, denoted by $\pi_{\mathrm{b}}(\Sigma)$, which is the set of vertex sets of balanced components of Σ. Note that $\pi_{\mathrm{b}}(\Sigma)$ is merely a partial partition of the vertices, meaning a partition of a subset of V. Clearly, $c(\Sigma) = |\pi(\Sigma)$ and $b(\Sigma) = |\pi_{\mathrm{b}}(\Sigma)|$.

Given Σ and a set of edges $S \subset E$, we understand $\pi_{\mathrm{b}}(S)$ to mean $\pi_{\mathrm{b}}(V, S, \sigma|_S)$; and similarly for π.

Contraction Now we can define contraction for an edge set $S \subseteq E$. Recall for ordinary graphs Γ/S has vertex set $V(\Gamma/S) = \pi(V, S) = \pi(S)$ and edge set $E(\Gamma/S) = E \setminus S$. For signed graphs, it turns out we want the vertex set to be $V(\Sigma/S) = \pi_{\mathrm{b}}(S)$.

Notes for 8 Feb. 2017 – Ted Ofner.

This lecture covered the definition and basic properties of contraction in signed graphs. We begin with the "balanced partial partition," a key object in the contraction process. For the duration of these notes Σ is a signed graph with vertex set V, edge set E, endpoint mapping $V : E \to \mathscr{P}(V)$ and signature $\sigma : E \to \{+,-\}$.

Definition 19. Let $S \subseteq E$; let $\Sigma|S$ be the signed graph $(V, S, end|_S, \sigma|_S)$. Let $B_S \subseteq \Sigma|S$ be the union of the balanced components of $\Sigma|S$. Then the *balanced partial partition* of S is the set

$$\pi_{\mathrm{b}}(S) = \pi(B_S) = \{V(B) \mid B \text{ is a balanced component of } \Sigma|S\}.$$

We can now define the contraction of Σ by S.

Definition 20. Let $S \subseteq E$. The *contraction* Σ/S is the graph with vertex set $\pi_{\mathrm{b}}(S)$, edge set $E \setminus S$, endpoint mapping $e \mapsto \{W \in \pi_{\mathrm{b}}(S) \mid V(e) \cap W \neq \varnothing\}$, and signature $\sigma|_{E \setminus S}$.

In essence, we normally contract all balanced portions of S, and delete all the unbalanced portions. This may create half or loose edges. One key property of the contraction is it's good behavior with respect to signature switching, which is captured by this proposition.

Proposition 24. *Let $\zeta : \pi_B(S) \to \{+,-\}$ be a state of Σ/S. Define the state $\tilde{\zeta}$ of Σ by*

$$\tilde{\zeta}(v) = \begin{cases} \zeta(w) & \text{if } v \in w \in \pi_{\mathrm{b}}(S), \\ \tilde{\zeta}(v) = + & \text{otherwise.} \end{cases}$$

Then $(\Sigma^{\tilde{\zeta}})/S = (\Sigma/S)^{\zeta}$.

Proof. Since switching by a state doesn't affect balance or imbalance of a graph, $(\Sigma^{\tilde{\zeta}})/S$ and $(\Sigma/S)^{\zeta}$ have the same vertex set $\pi_{\mathrm{b}}(S, \sigma) = \pi_{\mathrm{b}}(S, \sigma^{\tilde{\zeta}})$. Trivially, they have the same edge set, $E \setminus S$. Since the endpoint mapping does not depend on the signature, it is also unaffected. The only thing left to check is that the edge signatures match.

Let $e \in E \setminus S$. First, consider e as an edge of $\Sigma^{\tilde{\zeta}}$. If its ends u and v lie in B_S, let $w_u, w_v \in \pi_{\mathrm{b}}(S)$ be the components of B_S such that $u \in w_u, v \in w_v$. Then

$$\begin{aligned} \sigma^{\tilde{\zeta}}(e) &= \tilde{\zeta}(u)\sigma(e)\tilde{\zeta}(v) \\ &= \zeta(w_u)\sigma(e)\zeta(w_v) = \sigma^{\zeta}(e). \end{aligned}$$

Otherwise, e is a half or loose edge in Σ/S. The sign of a half or loose edge does not change the balance and unbalance of Σ/S, so we need not consider the sign of e. $\square$

Notes for 10 Feb. 2017 – Amelia Mattern.

Theorem 25. *Let S, T be disjoint sets of edges in Σ.*

(1) $(\Sigma \setminus S) \setminus T = \Sigma \setminus (S \cup T)$.
(2) $(\Sigma \setminus S)/T = (\Sigma/T) \setminus S$.
(3) $(\Sigma/S)/T = (\Sigma/T)/S = \Sigma/(S \cup T)$.

Proof. (1) Observe that $V((\Sigma \setminus S) \setminus T) = V(\Sigma) = V(\Sigma \setminus (S \cup T))$. Also,

$$E((\Sigma \setminus S) \setminus T) = (E(\Sigma) \setminus S) \setminus T = E(\Sigma) \setminus (S \cup T) = E(\Sigma \setminus (S \cup T)).$$

Finally, if $e \notin S \cup T$, then e remains unchanged in both $(\Sigma \setminus S) \setminus T$ and $\Sigma \setminus (S \cup T)$. Thus, $(\Sigma \setminus S) \setminus T = \Sigma \setminus (S \cup T)$.

(2) First observe that $V(\Sigma \setminus S) = V(\Sigma)$ and $\pi_{\mathrm{b}}(\Sigma|T) = \pi_{\mathrm{b}}((\Sigma \setminus S)|T)$ since they are the same signed graph. So $V((\Sigma \setminus S)/T) = V(\Sigma/T) = V((\Sigma/T) \setminus S)$. Furthermore, $E((\Sigma \setminus S)/T) = E \setminus (S \cup T)$. Since an edge in $S \cup T$ disappears from $(\Sigma/T) \setminus S)$ and $(\Sigma \setminus S)/T$, we need only consider what happens to $e \in E(\Sigma) \setminus (S \cup T)$. Let $e \in E(\Sigma) \setminus (S \cup T)$. The sign of the edge carries over when doing deletion or contraction, so we need not be concerned with the sign of e.

Case 1: Let $e{:}uv$ be a positive link. Note that in $\Sigma \setminus S$ there is no change to e.

Case 1.1: Suppose u and v are in the same component of $\Sigma|T$, call it T_1. There are two subcases.

If T_1 is balanced, then e becomes a positive loop in $(\Sigma \setminus S)/T$ and in $(\Sigma/T) \setminus S$.

If T_1 is unbalanced, then e becomes a loose edge in $(\Sigma \setminus S)/T$ and in $(\Sigma/T) \setminus S$.

Case 1.2: Suppose u and v are in different components of $\Sigma|T$. Let $u \in T_1$ and $v \in T_2$. There are four subcases to consider.

If both T_1 and T_2 are balanced, then e becomes $e{:}V(T_1)V(T_2)$ in $(\Sigma \setminus S)/T$ and in $(\Sigma/T) \setminus S$.

If both T_1 and T_2 are unbalanced, then e becomes a loose edge in $(\Sigma \setminus S)/T$ and in $(\Sigma/T) \setminus S$.

If T_1 is balanced and T_2 is unbalanced, then e becomes the half edge $e{:}V(T_1)$ in $(\Sigma \setminus S)/T$ and in $(\Sigma/T) \setminus S$.

If T_1 is unbalanced and T_2 is balanced, then e becomes the half edge $e{:}V(T_2)$ in $(\Sigma \setminus S)/T$ and in $(\Sigma/T) \setminus S$.

If e is a negative link, we switch so that it is positive and Case 1 applies.

Case 2: Suppose $e{:}vv$ is a loop or $e{:}v$ is a half edge. Then in $\Sigma \setminus S$ there is no change to e.

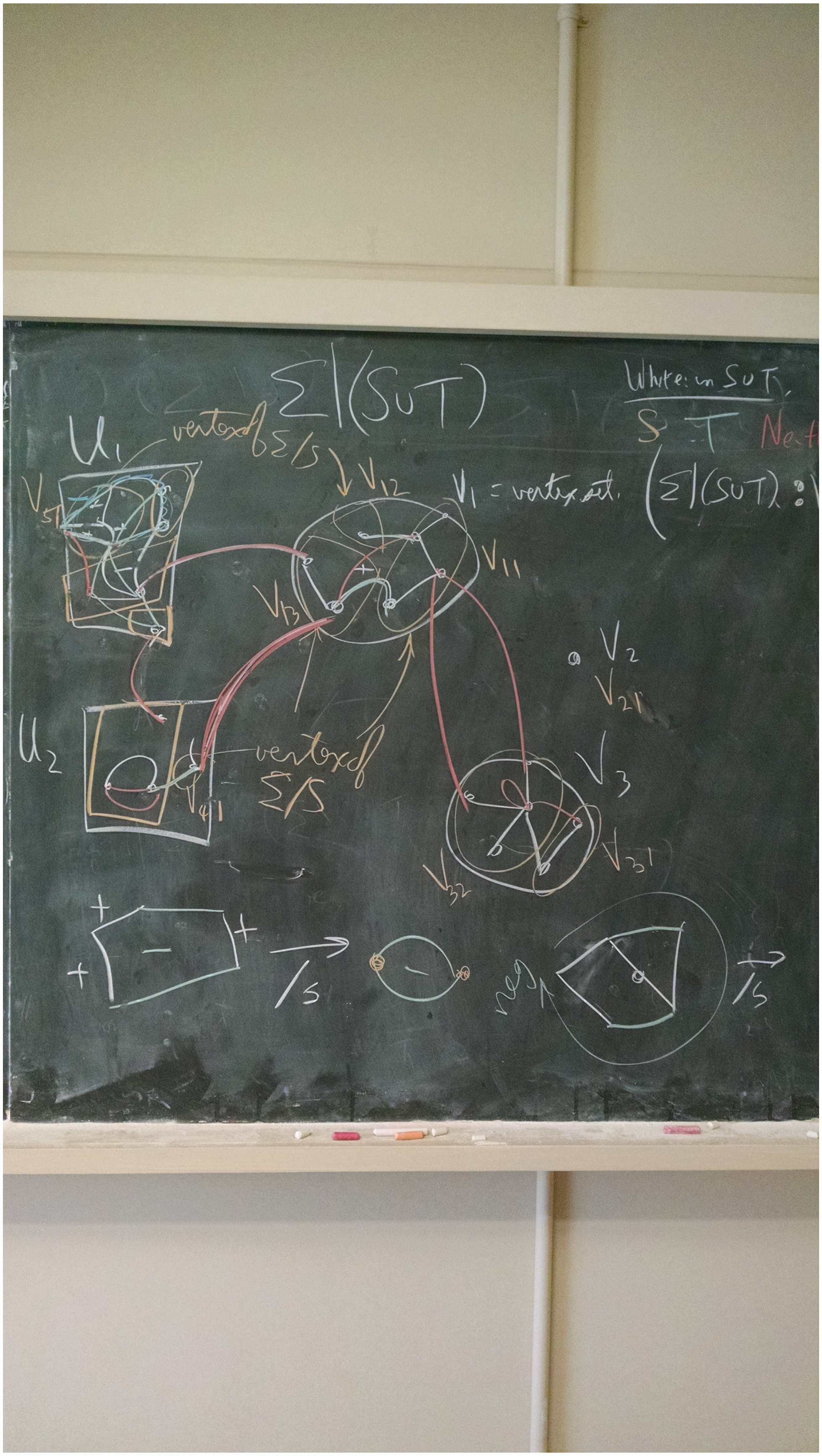
Case 2.1: Suppose v is in a balanced component of $\Sigma|T$, call it T_1. Then e becomes a loop $e{:}V(T_1)V(T_1)$ or half edge $e{:}V(T_1)$, respectively, in $(\Sigma \setminus S)/T$ and in $(\Sigma/T) \setminus S$.

Case 2.2: Suppose v is in an unbalanced component of $\Sigma|T$, call it T_1. Then e becomes a loose edge in $(\Sigma \setminus S)/T$ and in $(\Sigma/T) \setminus S$.

Case 3: Suppose $e{:}\varnothing$ is a loose edge. Then e remains unchanged in $(\Sigma \setminus S)/T$ and in $(\Sigma/T) \setminus S$.

Therefore $(\Sigma \setminus S)/T = (\Sigma/T) \setminus S$.

(3) First observe that $V(\Sigma/S) = \pi_{\mathrm{b}}(\Sigma|S)$ so $V((\Sigma/S)/T) = \pi_{\mathrm{b}}((\Sigma/S)|T)$. Also, $V(\Sigma/(S \cup T)) = \pi_{\mathrm{b}}(\Sigma|(S \cup T))$. We want to show that these are equal. By a previous result, all switching can be done before contraction and can be done so every balanced component of $\Sigma|S$ is all positive and consequently every balanced component of $\Sigma|(S \cup T)$ is all positive. For the remainder of the "proof" see the picture and notes below.


Σ|(S∪T)
White: in S∪T.
S T
U1
vertex of Σ/S
V12
V1 = vertex set.
(Σ|(S∪T) :
V51
V11
V13
V2
V21
U2
vertex of Σ/S
V41
V3
V31
V32
/S
neg
/S

We have $V_{1i}, V_{2i}, \ldots \in \pi_{\mathrm{b}}(\Sigma|S)$. Also, each $V_i \in \pi_{\mathrm{b}}(\Sigma|(S \cup T))$ is the union of some $V_{i1}, V_{i2}, \ldots \in \pi_{\mathrm{b}}(\Sigma|S)$. When we contract to $(\Sigma/S)/T$ we find that $\{V_{i1}, V_{i2}, \ldots\} \in \pi_{\mathrm{b}}((\Sigma/S)|T)$. So $\{V_{i1}, V_{i2}, \ldots\} \in V((\Sigma/S)/T)$. Also, $V_i \in V(\Sigma/(S \cup T))$. So each $V_i \in V(\Sigma/(S \cup T))$ is the union of some vertices of $V(\Sigma/S)$. So we are treating the set of V_i's and the union of the $V_i's$ as the same thing. Note that a negative circle in $S \cup T$, which doesn't exist in S, then becomes a half edge or a negative circle in $(\Sigma/S)|T$. The rest of the cases are left as an exercise. $\square$

Notes for 15 Feb. 2017 – Ted Ofner.

6. Closure

This lecture began with a discussion of the lattice correspondence between signed graphs and hyperplane arrangements from yesterday, but it quickly became clear some legwork was missing; namely, the definition of closure of an edge set. For the purpose of organization, I present the discussion of closure.

The Closure of an Edge Set in a Signed Graph.

Definition 21. Given $S \subseteq E$, define

$$V_u(S) := \{\text{vertices of the unbalanced components of S}\}$$

and

$$\mathrm{bcl}(S) := S \cup \{\text{loose edges}\} \cup \{e \notin S \mid \text{there exists a positive circle } C \text{ with } C \setminus e \subseteq S\};$$

this is called the *balance-closure* of S. (Not the "balanced closure"; it may not be balanced.) The *closure* of S is the edge set

$$\mathrm{clos}(S) := E{:}V_u(S) \cup \mathrm{bcl}(S).$$

Proposition 26 (Properties of Closure)**.** *If $S \subseteq E$ is balanced, then*

(1) $\mathrm{clos}(S) = \mathrm{bcl}(S)$.
(2) $\mathrm{bcl}(\mathrm{bcl}(S)) = \mathrm{bcl}(S)$.

Proof. (1) Since S is balanced, $V_u(S)$ is empty, thus $\mathrm{clos}(S) = \mathrm{bcl}(S)$.

(2) Switch Σ so $S \subseteq E^+$. Then

$$\begin{aligned}
\mathrm{bcl}(S) &= S \cup \{\text{loose edges}\} \\
&\qquad \cup \{e \notin S \mid \text{there is a path in } S \text{ connecting the endpoints of } e \text{ and } e \text{ is positive}\} \\
&= S \cup \{\text{loose edges}\} \\
&\qquad \cup \{e \in E^+ \mid \text{there is a path in } S \text{ connecting the endpoints of } e\}.
\end{aligned}$$

Since $\mathrm{bcl}(S) \subseteq E^+$, $\mathrm{bcl}(S)$ is balanced. Any circle in $\mathrm{bcl}(S)$ can have its non-S edges replaced by paths in S leaving a circle in S. [**What does "leaving" mean here?**] Thus any edge in $\mathrm{bcl}(\mathrm{bcl}(S))$ is in $\mathrm{bcl}(S)$. [**That proof is wrong. One doesn't get a circle.**] $\square$

At this point we stopped to prove that $\mathrm{clos}(S)$ was a closure operator, which gives us a lattice consisting of the closed sets in $\mathscr{P}(E)$.

Proposition 27. clos *is an abstract closure operator, i.e.,*

(1) $S \subseteq \mathrm{clos}(S)$,
(2) $\mathrm{clos}(\mathrm{clos}(S)) = \mathrm{clos}(S)$,
(3) *If* $S \subseteq Q$, *then* $\mathrm{clos}(S) \subseteq \mathrm{clos}(Q)$.

Proof. (1) $S \subseteq \mathrm{bcl}(S) \subseteq \mathrm{clos}(S)$.

(2) Let $e \in \mathrm{clos}(\mathrm{clos}(S))$. Then both endpoints of e lie in $V(\mathrm{clos}(S))$. [**This is the wrong statement.**] If both endpoints of e are in $\mathrm{bcl}(S)$, [**This is the wrong hypothesis.**] then $e \in \mathrm{bcl}(\mathrm{bcl}(S)) = \mathrm{bcl}(S)$. [**THIS IS WRONG.** $e \in \mathrm{bcl}(\mathrm{bcl}(S)) = \mathrm{bcl}(S)$ **is only valid if** S **is balanced.**]
If both endpoints of e are in $V_u(\mathrm{clos}(S))$, then $e \in V_u(S)$. Otherwise, $e \in \mathrm{bcl}(E : V_u(S)) \subseteq E{:}V_u(S)$.

(3) This is clear from inspecting the definitions. □

Notes for 13 Feb. 2017 – Josh Carey.

7. The Hyperplane Arrangement

A *hyperplane* in $\mathbb{R}^n$ is the solution set of a linear equation. A linear equation $\mathbf{a} \cdot \mathbf{x} = b$ is consistent if $\mathbf{a} \neq \mathbf{0}$ or $b = 0$, homogeneous if $b = 0$, and inhomogeneous if $b \neq 0$; we apply the same names to their solution sets.

An *arrangement of hyperplanes* is a finite set of hyperplanes. An arrangement of any hyperplanes, not necessarily homogeneous is *affine*; an arrangement of homogeneous hyperplanes is *homogeneous* or *linear* (and its hyperplanes are called the same). A *region* of $\mathscr{A}$ is a component of $\mathbb{R}^n \setminus (\bigcup \mathscr{A})$.

Notice that I allowed the equation $\mathbf{0} \cdot \mathbf{x} = 0$, whose solution set is $\mathbb{R}^n$, because it does arise naturally in operating with hyperplane arrangements; I call it the *degenerate hyperplane*. If the degenerate hyperplane belongs to an arrangement, then the arrangement has no regions, because its union is all of $\mathbb{R}^n$.

The *characteristic polynomial* of a hyperplane arrangement $\mathscr{A}$ is

$$p_{\mathscr{A}}(y) = \sum_{\mathscr{S} \subseteq \mathscr{A} :\, \bigcap \mathscr{S} \neq \varnothing} (-1)^{|\mathscr{S}|} y^{\dim(\bigcap \mathscr{S})}.$$

As a very special case, suppose $h_0 \in \mathscr{A}$ is the degenerate hyperplane. Then

$$p_{\mathscr{A}}(y) = \sum_{\substack{\mathscr{S} \subseteq \mathscr{A} \setminus \{h_0\} \\ \bigcap \mathscr{S} \neq \varnothing}} (-1)^{|\mathscr{S}|} y^{\dim \bigcap \mathscr{S}} + \sum_{\substack{\mathscr{S} \subseteq \mathscr{A} \setminus \{h_0\} \\ h_0 \cap (\bigcap \mathscr{S}) \neq \varnothing}} (-1)^{|\mathscr{S}|+1} y^{\dim(h_0 \cap (\bigcap \mathscr{S}))} = 0.$$

Here is a preview of the geometry of signed graphs (though no signed graphs are mentioned yet):

Theorem 28 (Zaslavsky [14])**.** *The number of regions of* $\mathscr{A}$ *equals* $(-1)^n p_{\mathscr{A}}(-1)$.

We define the *intersection semilattice* of $\mathscr{A}$ by

$$\mathscr{L}(\mathscr{A}) := \{\bigcap \mathscr{S} : \mathscr{S} \subseteq \mathscr{A} \text{ and } \bigcap \mathscr{S} \neq \varnothing\}.$$

the set of nonempty intersections of subsets of $\mathscr{A}\}$, partially ordered by reverse inclusion. This is a meet semilattice; that is, it has meets but not necessariiy joins. The problem with joins is that the join $t_1 \vee t_2$ of subspaces $t_1, t_2 \in \mathscr{L}(\mathscr{A})$ ought to be $t_1 \cap t_2$ but that is not available if the intersection is empty. However, a homogeneous arrangement never has this problem since every hyperplane contains the zero vector; and signed-graphic hyperplanes are homogeneous. Therefore, we shall be referring to the *intersection lattice*.

Now we define the hyperplanes that correspond to edges of a signed graph. Assume the graph has $V = \{v_1, \ldots, v_n\}$; the hyperplanes are in $\mathbb{R}^n$. Define

$$\begin{aligned} h^+_{ij} &:= (0, \ldots, 1, 0, \ldots, -1, 0, \ldots)^\perp = \{\mathbf{x} : x_i = x_j\}, \\ h^-_{ij} &:= (0, \ldots, 1, 0, \ldots, 1, 0, \ldots)^\perp = \{\mathbf{x} : x_i = -x_j\}, \\ h_i &:= \{\mathbf{x} : x_i = 0\}. \end{aligned}$$

For a signed graph Σ, we have the following bijection (with multiplicity; multiple edges of the same sign give the same geometrical hyperplane but we regard them as separate copies):

$$\begin{aligned} e{:}v_i v_j &\longleftrightarrow h^{\sigma(e)}_{ij} : x_i = \sigma(e) x_j \text{ (or } x_j = \sigma(e) x_i), \\ e{:}v_i &\longleftrightarrow h_i : x_i = 0, \\ e{:}\varnothing &\longleftrightarrow \text{the degenerate hyperplane } 0 = 0. \end{aligned}$$

Note that subscript order is immaterial since $h^\pm_{ij} = h^\pm_{ji}$. A positive loop, like a loose edge, corresponds to $h^+_{ii} = \mathbb{R}^n$ since its equation is $x_i = x_i$. A negative loop, like a half edge, corresponds to a coordinate hyperplane $x_i = 0$ since its hyperplane is $h^-_{ii} : x_i = -x_i$.

If we treat an unsigned graph Γ as all positive we have a bijection between edges and hyperplanes given by $e_{ij} \leftrightarrow h^+_{ij}$. This is how an unsigned graph would naturally be represented by a hyperplane arrangement (according to my advisor Curtis Greene); we believe it is correct because the correspondence extends to several other properties of a graph (as we shall see later in the course). Subsequently I proved these correspondences extend to signed graphs. (There has been much more work on graph-like arrangements but that takes us outside signed graphs.)

A *classical root system arrangement* is one of the following:

$$\begin{aligned} \mathscr{A}_{n-1} &= \{h^+_{ij} \mid i < j\}, \\ \mathscr{D}_n &= \{h^+_{ij}, h^-_{ij} \mid i < j\}, \\ \mathscr{B}_n = \mathscr{C}_n &= \mathscr{A}_{n-1} \cup \{h^-_{ij} \mid i \leq j\}. \end{aligned}$$

Thus, $\mathscr{H}[\Sigma] \subseteq \mathscr{B}_n$ for any signed graph.[7]

Here is a preview of two more theorems we'll prove:

Theorem 29. *There is a lattice isomorphism* $\mathscr{L}(\mathscr{H}[\Sigma]) \cong \operatorname{Lat} \Sigma$.

Deletion in Σ *and* $\mathscr{H}[\Sigma]$ *correspond, and contraction in* Σ *corresponds to induction in* $\mathscr{H}[\Sigma]$*; that is,* $\mathscr{H}[\Sigma \setminus S] = \mathscr{H}[\Sigma] \setminus \mathscr{H}[\Sigma|S]$ *and* $\mathscr{H}[\Sigma/S]$ *is the arrangement in* $\bigcap \mathscr{H}[\Sigma|S]$ *induced by intersection with* $\mathscr{H}[\Sigma \setminus S]$.

Theorem 30. *For a signed graph* Σ, $p_{\mathscr{H}[\Sigma]}(y) = \chi_\Sigma(y)$. *Thus,* $\mathscr{H}[\Sigma]$ *has* $(-1)^n \chi_\Sigma(-1)$ *regions.*

---

[7] I got into signed graphs because Richard Stanley asked me if I could treat root system arrangements by means of Theorem 28. I had to invent signed graphs to do it. For the result see [15].

I was not the first nor the second, nor probably the third or fourth, to invent signed graphs. The list of independent inventors has become fairly long; but I am confident from extensive study of the literature that Harary was first.

Notes for 17 Feb. 2017 – Chris Eppolito.

Let $\Sigma = (V, E, \sigma)$ be a signed graph with loose edge set $E^0 \subseteq E$. First we recall the definition of the closure operator clos_Σ.

Definition 22. The *balance-closure operator*[8] on Σ is defined by

$$\begin{aligned} \mathrm{bcl}_\Sigma &: \mathscr{P}(E) \to \mathscr{P}(E) \\ &: S \mapsto S \cup E^0 \cup \{e \in E \mid \text{there is a } C \in \mathscr{B} \text{ with } e \in C \text{ and } C \setminus e \subseteq S\} \end{aligned}$$

The *unbalanced vertex set* of Σ is defined by

$$V_u(\Sigma) = \{v \in V \mid v \text{ is in an unbalanced component of } \Sigma|S\}$$

The *closure operator* on Σ is given by

$$\mathrm{clos}_\Sigma : \mathscr{P}(E) \to \mathscr{P}(E) : S \mapsto \mathrm{bcl}(S) \cup (E{:}V_u(S))$$

Recall that $\mathrm{clos} = \mathrm{clos}_\Sigma$ is an abstract closure on E; the lattice of closed sets of E under clos is denoted by $\mathrm{Lat}(\Sigma)$. Moreover, $\mathrm{Lat}(\Sigma)$ has meet and join given by:

$$A \wedge B = A \cap B, \qquad\qquad A \vee B = \mathrm{clos}(A \cup B)$$

This fact follows from basic properties of abstract closure operators.

For ease of notation we let $\mathscr{L} := \mathscr{L}(\mathscr{H}[\Sigma])$ and $\mathscr{H} := \mathscr{H}[\Sigma]$.

We make the following simple observation.

Lemma 31 (Walk Lemma)**.** *Let $S \subseteq E$ be given. If W is a walk from u to v entirely in S, then $x_u = \sigma(W)x_v$ for all $x \in \alpha(S)$.*

This lemma essentially proves itself; if one wished to be overly formal, we could prove this result by mathematical induction on the length of the walk W.

Lemma 32 (Coordinates Lemma)**.** *Let $t \in \mathscr{L}$ and let B_i for $i \in [m]$ be the balanced components of $\beta(t)$. Switch so that all B_i are all positive. Let $\{V^\pm(B_i)\}$ be the Harary bipartition of B_i, where we choose $V^+(B_i) \neq \varnothing$. Then t is given by the following equations:*

$$\begin{aligned} &x_i = 0 && \text{for all } v_i \in V_u(\beta(t)), \\ &x_i = \begin{cases} c_j(x) & \text{for } v_i \in V^+(B_j) \\ -c_j(x) & \text{for } v_i \in V^-(B_j) \end{cases} && \text{for all } v_i \in B_j \text{ some constant } c_j(x). \end{aligned}$$

Proof of Lemma. First we show that t satisfies the equations above. Let $x \in t$ be arbitrary.

If $v_i \in V_u(\beta(t))$, then either the component K of v_i in $\Sigma|\beta(t)$ has a half-edge or v_i is on a negative circle. If $v_j \in K$ has a half-edge, then choosing any walk W from v_j to v_i gives the result by the Walk Lemma and the equation $x_j = 0$. Otherwise v_i is on a negative circle C. Let W be the walk in K beginning at v_i which follows C once to end at v_i; by construction $\sigma(W) = -$, so by the Walk Lemma $x_i = -x_i$; in particular $x_i = 0$.

If $v_i \in B_j$, then choose any vertex $v_+ \in V^+(B_i)$. Now let W be any walk from v_i to v_+ in B_j; by balance every such walk has the same sign. In particular $\sigma(W) = +$ precisely when $v_i \in V^+(B_j)$, and the Walk Lemma together with connectedness of B_j gives that for all there is a $c_j(x)$ such that $x_i = c_j(x)$ if $v_i \in V^+(B_j)$ and $x_i = -c_j(x)$ if $v_i \in V^-(B_j)$.

Hence we have shown that t satisfies the given equations.

[8]Not "balanced closure"; $\mathrm{bcl}(S)$ is not balanced in general.

On the other hand, if $x \in \mathbb{R}^V$ satisfies the above equations for some $t \in \mathscr{L}$, then the relations of t are a subset of the relations given above, which immediately implies that $x \in t$.

We conclude that the original statement is true. $\square$

We have the following proposition relating the hyperplane arrangement of Σ and the lattice of closed sets of E:

Proposition 33. *The functions α and β defined by*

$$\alpha : \mathscr{P}(E) \to \mathscr{L}(\mathscr{H}[\Sigma]) : S \mapsto \bigcap_{e \in S} h_e,$$
$$\beta : \mathscr{L}(\mathscr{H}[\Sigma]) \to \mathscr{P}(E) : t \mapsto \{e \in E \mid h_e \supseteq t\}$$

satisfy $\beta\alpha = \operatorname{clos}_\Sigma$, $\alpha\beta = \operatorname{id}_{\mathscr{L}}$, and $\operatorname{Im}(\beta) = \operatorname{Lat}(\Sigma)$.

We will prove the proposition via a series of lemmas.

Lemma 34. *$\beta(t)$ is closed (under $\operatorname{clos}_\Sigma$) for all $t \in \mathscr{L}$.*

Proof of Lemma. [**First we show that** $\operatorname{bcl}(\beta(t)) \subseteq \beta(t)$ **for all** $t \in \mathscr{L}$**.**] Let $e{:}uv$ be an edge such that $\beta(t) \cup e$ contains a positive circle C through e. Then $\sigma(e) = \sigma(C \setminus e)$, so by the Coordinates Lemma and the Walk Lemma we have $t \subseteq \bigcap_{a \in C \setminus e} h_a = \bigcap_{a \in C} h_a$; in particular $t \subseteq h_e$, so $e \in \beta(t)$; loose edges are trivially in $\beta(t)$ as they correspond to the trivial hyperplane. Hence $\operatorname{bcl}(\beta(t)) \subseteq \beta(t)$ as desired.

Next we show that $E{:}V_u(\beta(t)) \subseteq \beta(t)$ for all $t \in \mathscr{L}$. Note primarily that[9]

$$\begin{aligned} V_u(\beta(t)) &= \{v \in V \mid v \text{ is in an unbalanced component of } \beta(t)\} \\ &= \{v \in V \mid x_v = 0 \text{ for all } x \in t\} \end{aligned}$$

In particular every edge $e \in E{:}V_u(\beta(t))$ satisfies the conditions of the Coordinates Lemma; thus $t \subseteq h_e$ so $e \in \beta(t)$.

Hence $\operatorname{clos}(\beta(t)) = \operatorname{bcl}(\beta(t)) \cup E{:}V_u(\beta(t)) \subseteq \beta(t) \subseteq \operatorname{clos}(\beta(t))$ gives $\beta(t) \in \operatorname{Lat}(\Sigma)$. $\square$

Lemma 35. *The identity $\beta\alpha = \operatorname{clos}$ holds.*

Proof of Lemma. Let $S \subseteq E$ be arbitrary. Formulaically we have

$$\beta(\alpha(S)) = \beta\big(\bigcap_{s \in S} h_s\big) = \big\{e \in E \mid h_e \supseteq \bigcap_{s \in S} h_s\big\}$$

so we see $S \subseteq \beta\alpha(S)$; by our work above this gives $\operatorname{clos}(S) \subseteq \beta(\alpha(S))$. If $e \in \beta(\alpha(S))$, then $h_e \supseteq \bigcap_{s \in S} h_s = \alpha(S)$ so $\alpha(S)$ satisfies the equations of h_e; in particular one can deduce the equation of h_e from the equations of the hyperplanes h_s for $s \in S$. If e is a loose-edge or a positive loop, this trivially follows as h_e is the trivial hyperplane. If $e{:}v$ is a half-edge or $e{:}vv$ is a negative loop, then $x_v = 0$ implies that $v \in V_u(S)$, so $e \in E{:}V_u(S) \subseteq \operatorname{clos}(S)$. Otherwise $e{:}uv$ is a link. In this case there is a minimal set $S_e \subseteq S$ such that the equation for h_e may be deduced from those of $\{h_s \mid s \in S_e\}$.

We now show that S_e is a path in $|\Sigma|$. Note primarily that S_e determines a connected subgraph of $|\Sigma|$; if it were not, then either S_e is not minimal (i.e. contains an extra edge) or S_e does not connect the ends of e (i.e. does not determine the equation for h_e). Now if S_e is not a path, then S_e is not minimal as we may exclude any extraneous edges; thus S_e is a path subgraph from u to v in $\Sigma|S$. Hence $S_e \cup e$ is a circle in Σ.

[9]This is easily proved using the Walk Lemma.

Let W denote a walk from u to u following $S_e \cup e$ in cyclic order. (There are two such walks; both have the same sign, so for our purposes either will do.) If $\sigma(W) = -$, then $u, v \in V_u(S)$ giving that $e \in E{:}V_u(S) \subseteq \mathrm{clos}(S)$. If $\sigma(W) = +$, then $e \in \mathrm{bcl}(S) \subseteq \mathrm{clos}(S)$.

Hence in all cases $\beta(\alpha(S)) \subseteq \mathrm{clos}(S)$ and our above work gives $\beta(\alpha(S)) = \mathrm{clos}(S)$. As $S \subseteq E$ was arbitrary, we conclude that $\beta\alpha = \mathrm{clos}$. □

Lemma 36. *The functions α and β satisfy the identity $\alpha\beta = \mathrm{id}_{\mathscr{L}}$.*

Proof of Lemma. Let $t \in \mathscr{L}$. Now $\alpha(\beta(t)) = \bigcap_{e\in\beta(t)} h_e$ implies by definition of $\beta(t)$ that $t \subseteq \alpha(\beta(t))$. On the other hand $t = \bigcap_{s\in S} h_s$ for some subset $S \subseteq E$, so we immediately see that $S \subseteq \beta(t)$ and from this it follows that $\alpha(\beta(t)) \subseteq t$. Hence we have $\alpha\beta = \mathrm{id}$ as desired. □

The content of the proposition is precisely a conjunction of the previous three lemmas. Our main interest in this proposition is the following theorem.

Theorem 37. *As lattices, $\mathrm{Lat}(\Sigma)$ and $\mathscr{L}(\mathscr{H}[\Sigma])$ are isomorphic.*

The proof will be given in the next lecture.

Notes for 20 Feb. 2017 – Micah Loverro.

Recall $\mathscr{L} = \mathscr{L}(\mathscr{H}[\Sigma])$ and recall that we have the mappings

$$\alpha \colon \mathscr{P}(E) \to \mathscr{L}$$

and

$$\beta \colon \mathscr{L} \to \mathrm{Lat}(\Sigma).$$

We have shown, so far, that β is well defined.

Lemma 38. *Let $t \in \mathscr{L}$ and $S = \beta(t) \subseteq E$. Let $B_1, \ldots, B_m$ be the balanced components of S and $V_u(S)$ the vertices of unbalanced components. Then after switching so that each B_j is positive, t is given by*

$$\begin{cases} x_i = 0 & \text{for } v_i \in V_u(S), \\ x_i = c_j, & \text{a constant depending only on } j \text{, for all } v_i \in V(B_j). \end{cases}$$

Proof. Let $T = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x \text{ satisfies the equations of the lemma}\}$. Last time we showed $T \subseteq t$. Now we want to show $t \subseteq T$.

Recall that $\beta(t) = \{e \mid t \subseteq h_e\}$, and $\alpha(S) = \bigcap_{e\in S} h_e$.

Let's show that $T = \alpha(S)$. Let $x \in \alpha(S)$. Then, first of all, $x_i = 0$ if v_i supports a half edge or a positive loop.

Suppose S contains a circle $C = v_0 e_1 v_1 \cdots e_l v_l$ with $v_l = v_0$. Then $x_0 = \sigma(e_1)x_1 = \sigma(e_1)\sigma(e_2)x_2 = \cdots = \sigma(C)x_0$. If C is a negative circle, then $x_0 = 0$. If $V_u(S) \neq \varnothing$ then we have a vertex v_j with $x_j = 0$ in each component of $S{:}V_u(S)$. Suppose $x_i \in V(U_j)$ for an unbalanced component U_j. Then there is a path P from v_j to v_i which gives $x_i = \sigma(P)x_j$. Thus if $v_i \in V_u(S)$ we must have $x_i = 0$.

Now consider a balanced component B_j. If $v_i, v_h \in V(B_j)$ then there is a positive path P' connecting them, so $x_i = \sigma(P')x_h = x_h$. That is, x is constant on $V(B_j)$.

This proves that $\alpha(S) \subseteq T$.

For the other inclusion we follow the definitions. Let $x \in T$. Since $x_i = 0$ for every unbalanced vertex, we have $x \in h_e$ for each $e \in S{:}V_u(S)$. Since $x_i = x_h$ whenever v_i and v_h belong to the same balanced component, we have $x \in h_e$ for each e in the balanced component. That is, $x \in \alpha(S)$.

Now we know that $T = \alpha(S)$.

Finally, notice that $S = \beta(t) = \{e \mid h_e \supseteq t\} \leftrightarrow \{h_e \mid h_e \supseteq t\}$, so $\alpha(S) = \bigcap\{h_e \mid h_e \supseteq t\} = t$ since $t \in \mathscr{L}$ implies that t is the intersection of the hyperplanes that contain it.

So $t = \alpha(S) = T$. □

The next step to establishing our bijection is the following

Lemma 39.

1. $\beta(\alpha(S)) = \operatorname{clos}(S)$.
2. $\alpha(\beta(t)) = \mathrm{id}_{\mathscr{L}}(t)$.

[Part 2 duplicates a previous lemma. Compare proofs!]

Proof. For the first part, $\alpha(\beta(t)) = \bigcap\{h_e \mid e \in \beta(t)\} = \bigcap\{h_e \mid h_e \supseteq t\} = t$ as required.

For the second part, we begin with the observation that

$$\beta(\alpha(S)) = \{e \mid h_e \supseteq \alpha(S)\} = \left\{e \mid h_e \supseteq \bigcap \left\{h_f \mid f \in S\right\}\right\}.$$

We know from previously that $\beta(\alpha(S))$ is a closed set containing S, so $\beta(\alpha(S)) \supseteq \operatorname{clos}(S)$. It remains to show the reverse inclusion. That is, if $h_e \supseteq \bigcap\{h_f \mid f \in S\}$ then $e \in \operatorname{clos}(S)$.

Denote $\bigcap\{h_f \mid f \in S\}$ by t, and write $\operatorname{clos}(S) = \bigcup_{j=1}^{m} \operatorname{bcl}(B_j) \cup (E{:}V_u(S)) \cup \{\text{loose edges}\}$, from the definition. We may as well also assume (by switching) that all B_j are all positive.

If e joins two balanced components $B_j, B_{j'}$ then $h_e \not\supseteq t$ since the Coordinate Lemma gives no equations combining vertices of B_j with those of $B_{j'}$.

If $e{:}v_iv_h$ with v_i, v_h in the same B_j then $x_i = x_h$ so $e \in \operatorname{bcl}(B_j)$.

If $e{:}v_iv_h$ or $e{:}v_i$ with $v_i, v_h \in V_u(S)$ then clearly $e \in \operatorname{clos}(S)$.

In any case we have $\beta(\alpha(S)) = \operatorname{clos}(S)$. □

Now we have shown that β and $\alpha|_{\mathrm{Lat}(\Sigma)}$ are inverse bijections.

Notes for 22 Feb. 2017 – Amelia Mattern.

We continue our discussion of the characteristic polynomial of a hyperplane arrangement. Recall its definition:

$$p_{\mathscr{H}[\Sigma]}(\lambda) := \sum_{\mathscr{S} \subseteq \mathscr{H}[\Sigma]} (-1)^{|\mathscr{S}|} \lambda^{\dim(\bigcap \mathscr{S})} = \sum_{t \in \mathscr{L}(\mathscr{H}[\Sigma])} \mu(\varnothing, t) \lambda^{\dim t}$$

where

$$\mu(\varnothing, t) = \begin{cases} 0 & \text{if } \hat{0} \text{ is not closed,} \\ \mu(\hat{0}, t) = 1 & \text{if } \hat{0} \text{ is closed.} \end{cases}.$$

In the above statement the first equality is the definition and the second equality comes from a previous theorem.

In general, if we have a set X and an abstract closure operator $W \mapsto \overline{W}$ on X, then the class of closed sets, $\mathscr{C}$, is a lattice, and for $Z \in \mathscr{C}$ we define

$$\mu(Y, Z) = \begin{cases} 0 & \text{if } Y \notin \mathscr{C}, \\ \mu_{\mathscr{C}}(Y, Z) & \text{if } Y \in \mathscr{C}. \end{cases}$$

Lemma 40. *Let $\mathscr{C}$, μ, and Z be as above. Then*

$$\mu(Y, Z) = \sum_{\substack{Y \subseteq S \subseteq Z \\ \overline{S} = Z}} (-1)^{\#S}.$$

Remark 1. If we discard $X \setminus Z$ and Y from X we have a structure isomorphic to $[Y, Z]_{\mathscr{P}(X)}$ with closure $W \mapsto \overline{W \cup Y} \setminus Y$ and $\mu(Y, Z)$ equal to $\mu(\varnothing, Z \setminus Y)$.

Proof. (1) If $Y \notin \mathscr{C}$, then

$$\sum_{Y \subseteq S \subseteq Z\colon\ \overline{S} = Z} (-1)^{\#S} = \sum_{Y \subseteq S_1 \subseteq Y} \sum_{\substack{S_2 \subseteq Z \setminus Y \\ \overline{S_1 \cup S_2} = Z}} (-1)^{\#(S_1 \cup S_2)} = \sum_{Y \subseteq S_1 \subseteq \overline{Y}} (-1)^{\#S_1} \sum_{\substack{S_2 \subseteq Z \setminus \overline{Y} \\ \overline{S_1 \cup S_2} = Z}} (-1)^{\#S_2}.$$

But

$$\sum_{Y \subseteq S_1 \subseteq \overline{Y}} (-1)^{\#S_1} = (-1)^{\#Y} \sum_{T \subseteq \overline{Y} \setminus Y} (-1)^{\#T} = (-1)^{\#Y} 0^{|\overline{Y} \setminus Y|} = 0.$$

(2) If $Y \in \mathscr{C}$, then $(-1)^{\#Y} 0^{|\overline{Y} \setminus Y|} = 1$ and $S_1 = Y = \overline{Y}$. So

$$\sum_{\substack{Y \subseteq S \subseteq Z \\ \overline{S} = Z}} (-1)^{\#S} = \sum_{Y \subseteq S_1 \subseteq \overline{Y}} (-1)^{\#S_1} \sum_{\substack{S_2 \subseteq Z \setminus \overline{Y} \\ \overline{S_1 \cup S_2} = Z}} (-1)^{\#S_2} = (-1)^{\#Y} 0^{|\overline{Y} \setminus Y|} \sum_{\substack{S_2 \subseteq Z \setminus \overline{Y} \\ \overline{S_1 \cup S_2} = Z}} (-1)^{\#S_2}$$

$$= \sum_{\substack{S_2 \subseteq Z \setminus \overline{Y} \\ \overline{Y \cup S_2} = Z}} (-1)^{\#S_2} = (-1)^{\#Y} \sum_{\substack{Y \subseteq S \subseteq Z \\ \overline{S} = Z}} (-1)^{\#S}.$$

Define

$$F(Y, Z) := (-1)^{\#Y} \sum_{\substack{Y \subseteq S \subseteq Z \\ \overline{S} = Z}} (-1)^{\#S} \text{ and } G(Y, A) = \sum_{\substack{Y \subseteq Z \subseteq A \\ Y, A, Z \in \mathscr{C}}} F(Y, Z).$$

Then

$$G(Y, A) = \sum_{\substack{Y \subseteq S \subseteq Z \subseteq A \\ \overline{S} = Z}} (-1)^{\#Y} (-1)^{\#S} = \sum_{Y \subseteq S \subseteq A} (-1)^{|S \setminus Y|} = 0^{|A \setminus Y|}.$$

Therefore

$$G(Y, A) = \begin{cases} 0 & \text{if } Y \subset A, \\ 1 & \text{if } Y = A, \end{cases}$$

so

$$G(Y, A) = \sum_{Y \subseteq Z \subseteq A} \zeta_{\mathscr{C}}(Z, A) F(Y, Z),$$

where $\zeta_{\mathscr{C}}$ is the *zeta function* of $\mathscr{C}$, defined by

$$\zeta_{\mathscr{C}}(Y, A) := \begin{cases} 0 & \text{if } Y \subseteq A, \\ 1 & \text{if } Y \nsubseteq A. \end{cases}$$

By Möbius inversion in $\mathscr{C}$,

$$F(Y,A) = \sum_{Y\subseteq Z\subseteq A} \mu(Z,A)G(Y,Z) = \mu(Y,A)$$

since $\mu(Z,A) = 0$ unless $Z = A$. $\square$

Notes for 24 Feb. 2017 – Josh Carey.

8. Coloring

Here is the classical definition of graph coloring. Let $\Gamma = (V,E)$ be a graph. A *coloration* of Γ is a function $\kappa : V \to [k]$, where $k \in \mathbb{Z}_{\geq 0}$. ($[0]$ is defined to be the empty set.) It is *zero-free* if it never takes the value 0. A coloration is *proper* if for each edge $e{:}vw$ we have $\kappa(v) \neq \kappa(w)$.

Now the generalization to signed graphs. Let $\Sigma = (V,E,\sigma)$ be a signed graph. A *coloration* of Σ is a function $\kappa : V \to C_k$, where $C_k = \{-k,\dots,0,\dots,k\}$. A coloration of Σ is *proper* if for each link or loop $e{:}vw$, we have $\kappa(v) \neq \sigma(e)\kappa(w)$; for a half edge $e{:}v$, we have $\kappa(v) \neq 0$; and Σ has no loose edges.

Define $\chi_\Sigma(\lambda)$, for $\lambda \in \mathbb{Z}_{\geq 0}$, as the number of proper λ-colorations of Σ.

Theorem 41 (Birkhoff–Whitney)**.** *For a graph, the chromatic function $\chi_\Gamma(\lambda)$ is a monic polynomial of degree $n = \#V$.*

We are going to prove the same theorem for a signed graph. To begin, we show the chromatic function is invariant under switching.

Proposition 42. $\chi_{\Sigma^X}(\lambda) = \chi_\Sigma(\lambda)$ *for* $\lambda \in \mathbb{Z}_{\geq 0}$.

Proof. Let κ be a proper coloration of Σ. We want a corresponding proper coloration of Σ^X. To get it we switch κ to κ^X defined by

$$\kappa^X(v) = \begin{cases} -\kappa(v) \text{ if } v \in X, \\ \ \ \kappa(v) \text{ if } v \notin X. \end{cases}$$

Now we have a bijection, which is worth stating formally.

Lemma 43. *If $X \subseteq V(\Sigma)$, $\kappa \mapsto \kappa^X$ gives a bijection between proper colorations of Σ and proper colorations of Σ^X.*

This lemma is almost immediate (I leave it to the reader) and it implies our proposition. $\square$

Theorem 44 (Incorrect theorem)**.** *For any edge $e \in E(\Sigma)$, we have $\chi_\Sigma(\lambda) = \chi_{\Sigma\setminus e}(\lambda) - \chi_{\Sigma/e}(\lambda)$.*

Incomplete proof. We consider two cases.

Case 1: e is a loose edge or positive loop. Then $\Sigma/e = \Sigma \setminus e$ so $\chi_{\Sigma\setminus e}(\lambda) - \chi_{\Sigma/e}(\lambda) = 0$. Also, Σ has no proper colorations, thus $\chi_\Sigma(\lambda) = 0$.

Case 2: e is a negative loop or half edge at v. Fix λ. We want a bijection between $P(\Sigma\setminus e)$ and $P(\Sigma) \sqcup P(\Sigma/e)$. Let $\kappa \in P(\Sigma \setminus e)$. Either $\kappa(v) = 0$ or $\kappa(v) \neq 0$. In the former case, κ is proper on Σ but improper on Σ/e and in the latter case, κ is proper on Σ/e but improper on Σ. $\square$

[We discovered a mistake in the theorem and stopped until next class.]

Notes for 27 Feb.–1 Mar. 2017 – Chris Eppolito.

8.1. **Deletion-Contraction Formula for the Chromatic Quasipolynomial.**

Let $\Sigma = (V, E, \sigma)$ be a signed graph. We introduce the following notation for $\lambda \in \mathbb{Z}_{\geq 0}$:

$$\mathrm{K}_\Sigma(\lambda) := \{\kappa \mid \kappa \text{ is a proper } \lambda\text{-coloration of } \Sigma\}.$$

Recall that we define $\chi_\Sigma(\lambda) := \#\mathrm{K}_\Sigma(\lambda)$ and the color set for λ-coloring is denoted by $\mathbf{C}_\lambda$.

Proposition 45. *Each proper λ-coloration of Σ induces a proper λ-coloration of Δ for every $\Delta \subseteq \Sigma$. In particular. if Δ spans Σ, this gives an injection $\mathrm{K}_\Sigma(\lambda) \to \mathrm{K}_\Delta(\lambda)$.*

Proof of Proposition. Let κ be a proper λ-coloration of Σ. Then every edge of Δ is proper under κ, so the mapping

$$\kappa|_\Delta : V(\Delta) \to \mathbf{C}_\lambda : v \mapsto \kappa(v)$$

is a proper λ-coloration of Δ, as desired. It follows trivially **[NOT SO!]** from the definition of $\kappa|_\Delta$ that the mapping $\kappa \mapsto \kappa_{|\Delta}$ is injective. □

We are in the process of proving the following theorem. Recall that an edge is unbalanced if and only if it is a half edge or a negative loop.

Theorem 46 (Deletion-Contraction Formula)**.** *Let e be an edge of Σ. If either e is not an unbalanced edge or λ is odd, then $\chi_\Sigma(\lambda) = \chi_{\Sigma\setminus e}(\lambda) - \chi_{\Sigma/e}(\lambda)$.*

Proposition 47. *Let ζ be a switching of Σ. Every proper coloration κ of Σ induces a proper coloration κ^ζ of Σ^ζ via the formula $\kappa^\zeta(v) = \zeta(v)\kappa(v)$. Furthermore, the mapping $\hat{\zeta} : \kappa \mapsto \kappa^\zeta$ is an involutory bijection $\mathrm{K}_\Sigma(\lambda) \to \mathrm{K}_{\Sigma^\zeta}$ for all $\lambda \in \mathbb{Z}_{\geq 0}$.*

Proof. Suppose κ is a proper coloration of Σ; thus $|\Sigma|$ has no positive loops and no loose edges. Trivially for all half-edges $e{:}v$ and negative loops $e{:}vv$ of $|\Sigma|$ we have $\kappa^\zeta(v) = \zeta(v)\kappa(v) \neq 0$ as $\kappa(v) \neq 0$ by assumption. Now suppose $e{:}uv$ is a link; thus $\kappa(u) \neq \sigma(e)\kappa(v)$, and from this we derive the following desired non-equality:

$$\kappa^\zeta(u) = \zeta(u)\kappa(u) \neq \zeta(u)\sigma(e)\kappa(v) = (\zeta(u)\sigma(e)\zeta(v)) \cdot (\zeta(v)\kappa(v)) = \sigma^\zeta(e)\kappa^\zeta(v).$$

Hence κ^ζ is a proper coloration of Σ^ζ as desired.

A simple computation shows $(\kappa^\zeta)^\zeta = \kappa$, so this mapping is an involution as desired. □

Now, let $\tilde{\pi}_e$ denote the function taking a vertex in V to its component in the graph $(V, \{e\})$; that is, $\tilde{\pi}_e : V \to \pi(\Sigma|\{e\})$.

Lemma 48. *Let $e{:}uv$ be a positive link of Σ and let κ be a proper coloration of $\Sigma \setminus e$.*

(1) If $\kappa(u) = \kappa(v)$, then the following is a proper coloration of Σ/e:

$$\kappa_{/e} : V/_{u\sim v} \to \mathbf{C}_\lambda : \tilde{\pi}_e(x) \mapsto \kappa(x).$$

(2) If $\kappa(u) \neq \kappa(v)$, then the following is a proper coloration of Σ:

$$\kappa_e : V \to \mathbf{C}_\lambda : x \mapsto \kappa(x).$$

Proof. The proof is essentially a straightforward verification of conditions.

Let $e{:}uv$ be a positive link of Σ and let κ be a proper coloration of $\Sigma \setminus e$.

Part 1: If $\kappa(u) = \kappa(v)$, then immediately $\kappa_{/e}$ is well-defined. Conditions on half-edges and negative loops have not changed under $\kappa_{/e}$, so all such edges are proper. Notice that $\kappa(u) = \kappa(v)$ implies by propriety of κ that there are no other links between u and v. Hence it follows that $\kappa_{/e}$ is proper.

Part 2: If $\kappa(u) \neq \kappa(v)$, then as e is positive and κ proper on $\Sigma \setminus e$ it follows immediately that κ_e is a proper coloration of Σ.

We conclude that the statement holds. ☐

Lemma 49. *Let $e{:}uv$ be a positive link of Σ. If κ is a proper λ-coloration of Σ/e, then the function $\tilde{\kappa} : V \to \mathbf{C}_\lambda : x \mapsto \kappa(\tilde{\pi}_e(x))$ is a proper λ-coloration of $\Sigma \setminus e$.*

Proof. For vertices u, v that are adjacent in $\Sigma \setminus e$ by an edge of sign ε, $\pi_e(u)$ and $\pi_e(v)$ are adjacent in Σ/e by an edge or edges of the same sign. Therefore, $\tilde{\kappa}(u) = \kappa(\pi_e(u)) \neq \varepsilon\kappa(\pi_e(v)) = \varepsilon\tilde{\kappa}(v)$. Thus, $\tilde{\kappa}$ is proper. ☐

Proposition 50. *Let λ be odd and e a negative loop or half-edge in Σ. There is a bijection $\mathrm{K}_{\Sigma\setminus e}(\lambda) \leftrightarrow \mathrm{K}_\Sigma(\lambda) \sqcup \mathrm{K}_{\Sigma/e}(\lambda)$.*

Proof. Let λ be odd, and suppose e is either a negative loop or half-edge at the vertex v. Let κ be a proper λ-coloration of $\Sigma \setminus e$. ☐

Proposition 51. *Let e be a positive link of Σ and let $\lambda \in \mathbb{Z}_{\geq 0}$. There is a bijection*

$$\mathrm{K}_{\Sigma\setminus e}(\lambda) \leftrightarrow \mathrm{K}_\Sigma(\lambda) \sqcup \mathrm{K}_{\Sigma/e}(\lambda).$$

We could actually prove more. The bijection as described in the previous lemmas has a nice form for all λ. We only need the existence of a bijection to prove the theorem, so this is what I will prove.

Proof. Primarily, notice that the maps $\mathrm{K}_\Sigma(\lambda) \to \mathrm{K}_{\Sigma\setminus e}$ and $\mathrm{K}_{\Sigma/e}(\lambda) \to \mathrm{K}_{\Sigma\setminus e}$ have disjoint images. This follows trivially from the fact that all images κ of proper colorations of Σ/e have $\kappa(u) = \kappa(v)$ while all images of proper colorations of Σ necessarily have $\kappa(u) \neq \kappa(v)$ as $e{:}uv$ is positive. On the other hand, [**Proposition 45 gives that the former function is injective. That the latter function is injective follows trivially from its definition.**] that these two mappings are injective follow in straightforward fashion from their definitions. Hence gluing these maps we obtain an injective map $\mathrm{K}_\Sigma(\lambda) \sqcup \mathrm{K}_{\Sigma/e}(\lambda) \to \mathrm{K}_{\Sigma\setminus e}(\lambda)$.

Let two proper λ-colorations κ and ρ of $\Sigma \setminus e$ be given so that the mapping μ described in Lemma 48 has $\mu(\kappa) = \mu(\rho)$. As the image of a coloration under μ agrees on all vertices (up to identification) with its preimage, we may conclude that $\kappa = \rho$. Hence this mapping is injective.

We conclude that the proposition holds. ☐

Proof of Theorem. If e is a loose edge or a positive loop, then $\chi_\Sigma(\lambda) = 0$; on the other hand, $\Sigma \setminus e = \Sigma/e$ in this case, so the result holds.

If $e{:}uv$ is a link, choose any switching function ζ for Σ such that e is positive in Σ^ζ. What remains follows by straightforward applications of Proposition 47 and Proposition 51.

If $e{:}uv$ is a half-edge or negative loop and λ is odd, then Proposition 50 gives the result.

Hence the Proposition holds in all cases as desired. ☐

Theorem 52. *Suppose $\Sigma = (l \times \Sigma_0) \cup \bigsqcup_{i\in[k]} \Sigma_i$, where each Σ_i, for $i \in [m]$, is without loose edges and $(l \times \Sigma_0)$ denotes a graph consisting of l loose edges. Then $\chi_\Sigma = 0^l \prod_{i=1}^m \chi_{\Sigma_i}$.*

Proof. By Proposition 45, every coloration of Σ is a combination of colorations of its components Σ_i. Moreover, every coloration of Σ arises in this way. Furthermore, the coloration of Σ is proper iff the component colorations are proper and $l = 0$. Hence the result. $\square$

Recall that $b(S)$ is the number of balanced components of S in Σ for all $S \subseteq E$.

Theorem 53. *For $\lambda \in \mathbb{Z}_{\geq 0}$ we have the following formula:*

$$\chi_\Sigma(\lambda) = \begin{cases} \sum_{S\subseteq E} (-1)^{\#S} \lambda^{b(S)} & \text{if } \lambda \text{ is odd,} \\ \sum_{S\subseteq E,\ balanced} (-1)^{\#S} \lambda^{b(S)} & \text{if } \lambda \text{ is even.} \end{cases}$$

We begin the proof with an easy case.

Lemma 54. *There is equality in Theorem 53 if Σ has positive loops or loose edges.*

Proof. Let $p_\Sigma(\lambda)$ denote the right-hand side of the equation in the statement of the theorem. Let L denote the set of positive loops and loose edges of Σ. If $L \neq \varnothing$, then $\chi_\Sigma(\lambda) = 0$. On the other hand (combining two formulas in one by the condition "[balanced]" when λ is even):

$$\begin{aligned} p_\Sigma(\lambda) &= \sum_{\substack{S\subseteq E \\ [balanced]}} (-1)^{\#S} \lambda^{b(S)} = \sum_{S_1\subseteq L} (-1)^{\#S_1} \sum_{\substack{S_2\subseteq E\setminus L \\ [balanced]}} (-1)^{\#S_2} \lambda^{b(S_2)} \\ &= \Bigg(\sum_{S_1\subseteq L} (-1)^{\#S_1} \Bigg) \Bigg(\sum_{\substack{S_2\subseteq E\setminus L \\ [balanced]}} (-1)^{\#S_2} \lambda^{b(S_2)} \Bigg) = 0 \sum_{\substack{S_2\subseteq E\setminus L \\ [balanced]}} (-1)^{\#S_2} \lambda^{b(S_2)} = 0. \end{aligned}$$ $\square$

Proposition 55. *For a set $S \subseteq E(\Sigma)$, the following statements are equivalent:*

(1) S is balanced.
(2) $\mathrm{bcl}(S)$ is balanced.
(3) $\mathrm{clos}_\Sigma(S)$ is balanced.

Proof. The reverse implications (3) $\implies$ (2) $\implies$ (1) follow trivially as subsets of balanced sets are balanced. To see that (1) $\implies$ (3), switch Σ by ζ so that S is all positive. Note that $V_u(S) = \varnothing$, so $\mathrm{clos}(S) = \mathrm{bcl}(S)$ is necessarily a subset of the loose edges of Σ^ζ and the positive edges of Σ^ζ by its definition. As balance is preserved under switching, the result holds. $\square$

Proposition 56. *For $S \subseteq E$, $b(S) = b(\mathrm{clos}(S))$.*

Proof. By the definition of the closure operator, $b(\mathrm{clos}(S)) = b(\mathrm{bcl}(S))$. On the other hand, $b(\mathrm{bcl}(S)) = b(S)$ by Proposition 55. $\square$

Lemma 57. *A set $T \cup e$ is balanced in Σ if and only if T is balanced in Σ/e and $V(e)$ is not an unbalanced component of $\Sigma|(T \cup e)$.*

Proof. The sufficiency follows by switching $T \cup e$ to all positive and noticing that T is then all positive in Σ/e. For the necessity, suppose $T \cup e$ is unbalanced in Σ we consider several cases.

Case 0: If e is a positive loop or a loose edge, then $\Sigma \setminus e = \Sigma/e$, so the result holds trivially.

Case 1: If e is a link, then switch so e is positive. Thus $T \cup e$ contains a negative circle or a half-edge. The half-edge case implies T is not balanced in Σ/e. Otherwise $T \cup e$ contains a negative circle C. If $e \in C$, then $C/e \subseteq T$ is a negative circle in the contraction, so T is not balanced. If $e \notin C$, either e is a chord of C or not. If not, then $C \subseteq T$ is a negative circle of Σ/e. If so, then $C \cup e$ is a θ-graph, so we may write $C = P \sqcup Q$ where $P \cup e$ and $Q \cup e$ are circles of Σ. Now $\sigma(C) = \sigma(P)\sigma(Q) = -$, so exactly one of these is negative; by choice of notation $\sigma(P) = -$. In particular, $P \subseteq T$ is a negative circle in the contraction, so T is unbalanced in the contraction.

Case 2: If e is a half-edge or negative loop, consider $V(e)$. If $V(e)$ is an component of $\Sigma|(T \cup e)$, then this component vanishes in the contraction and T is thus balanced in the contraction if and only if T is balanced in Σ. Otherwise, either a half-edge or a loose edge is created in the contraction by e.

Thus we see that the original statement holds. □

An important corollary is a straightforward consequence of the proof above.

Corollary 58. *For $T \subseteq E$ and $e \in E \setminus T$, $b_\Sigma(T \cup e) = b_{\Sigma/e}(T)$.*

Lemma 59. *The right-hand side of the formula in the theorem satisfies the same deletion-contraction recursion as the left side.*

Proof. Let $p_\Sigma(\lambda)$ denote the right-hand side of the formula in the theorem. Let $e \in E$ be fixed. We compute as follows:

$$\begin{aligned}
p_\Sigma(\lambda) &= \sum_{\substack{S \subseteq E \\ [\mathit{balanced}]}} (-1)^{\#S} \lambda^{b_\Sigma(S)} = \sum_{\substack{S \subseteq E \setminus e \\ [\mathit{balanced}]}} (-1)^{\#S} \lambda^{b_\Sigma(S)} + \sum_{\substack{e \in S \subseteq E \\ [\text{balanced in } \Sigma]}} (-1)^{\#S} \lambda^{b_\Sigma(S)} \\
&= p_{\Sigma \setminus e}(\lambda) + \sum_{\substack{T \subseteq E \setminus e \\ [T \cup e \text{ balanced in } \Sigma]}} (-1)^{\#T+1} \lambda^{b_\Sigma(T \cup e)} \\
&= p_{\Sigma \setminus e}(\lambda) - \sum_{\substack{T \subseteq E \setminus e \\ [T \text{ balanced in } \Sigma]}} (-1)^{\#T} \lambda^{b_{\Sigma/e}(T)} \\
&= p_{\Sigma \setminus e}(\lambda) - p_{\Sigma/e}(\lambda)
\end{aligned}$$

Hence we conclude that the original statement is true. □

The proof of the following lemma is straightforward.

Lemma 60. *The right-hand side of the equation in the theorem is multiplicative over components.*

Proof of Theorem. Both the left-hand and right-hand sides of the equation satisfy the same base and recursion. □

Notes for 8 Mar. 2017 – Josh Carey.

We define the chromatic quasipolynomial $\chi_\Sigma(\lambda)$ for $\lambda \in \mathbb{Z}_{\geq 0}$. It will have the property that $\chi_\Sigma|_{\text{odd integers } \geq 0}$ is $\chi_\Sigma(\lambda)$ (the chromatic polynomial) and $\chi_\Sigma|_{\text{even integers } \geq 0}$ is a polynomial, $\chi^*_\Sigma(\lambda)$ (the zero-free chromatic polynomial).

First, we define a quasipolynomial. It is a kind of function of (some) integers. A function f on the positive, or nonegative, integers is a *quasipolynomial* if there are a number p and p polynomials, $f_0, f_1, \ldots, f_{p-1}$, such that $f(n) = f_{n \bmod p}(n)$.

Here is a big theorem in graph theory that we will extend to signed graphs.

Theorem 61 (Stanley)**.** *For a graph Γ, $(-1)^n\chi_\Gamma(-1)$ is the number of acyclic orientations of Γ.*

The general problem, necessary if one is to make use of the generalized Stanley theorem for example, is to find the chromatic quasipolynomial of a signed graph. Consider, for example, $+K_n$. Then $\chi_{+K_0}(\lambda) = 1$, $\chi_{+K_1}(\lambda) = \lambda$, $\ldots$, $\chi_{+K_n} = (\lambda - (n-1))\chi_{+K_{n-1}}$; so for arbitrary n, we have $\chi_{+K_n}(\lambda) = (\lambda)_n$ where $(\lambda)_n$ denotes the falling factorial:

$$(\lambda)_n = \begin{cases} \lambda(\lambda-1)\cdots(\lambda-[n-1]) & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

Here is another example where we can compute the chromatic polynomial explicitly. A graph Γ is *chordal* when it is simple and has no induced circles of length bigger than 3. Equivalently, Γ is chordal if there is a simplicial vertex ordering, $(v_1, \ldots, v_n)$ (this is a well-known theorem of G.A. Dirac). Letting $\Gamma_i := \Gamma{:}\{v_1, \ldots, v_i\}$, being a simplicial vertex ordering means the neighborhood $N(\Gamma_i; v_i)$ is a clique for all i. Now let $d_i = d(\Gamma_i; v_i)$, the degree of v_i in Γ_i (not its degree in Γ). Then $\chi_{+\Gamma}(\lambda) = (\lambda - d_n)\chi_{+\Gamma \setminus v_n}(\lambda) = \prod_{i=1}^n (\lambda - d_i)$.

You will have noticed that

Proposition 62. *For a balanced signed graph $\Sigma = (\Gamma, \sigma)$, $\chi_\Sigma(\lambda) = \chi_\Gamma(\lambda)$.*

Proof. If $\Sigma = +\Gamma$, the only requirement on a proper coloration is that $\kappa(u) \neq \kappa(v)$ if u, v are adjacent; so the proper colorations are the same for $+\Gamma$ as for Γ (we can use the signed color set $\mathbf{C}_\lambda$ for Γ; only the number of colors matters). If Σ is not all positive, it switches to all positive (Corollary 3) and that switching doesn't change the chromatic function (Proposition 42), so $\chi_\Sigma = \chi_{+\Gamma} = \chi_\Gamma$. □

Now an important connection to the hyperplane arrangement.

Theorem 63. *$\chi_\Sigma(\lambda) = p_{\mathscr{H}[\Sigma]}(\lambda)$ for odd integers λ, where $\chi_\Sigma(\lambda)$ is the chromatic polynomial.*

This is immediate because the two functions have the same algebraic formula.

The 0-colored vertices are all nonadjacent. Also, they cannot support any loops or half edges. We call such a set *stable*. Equivalently, $X \subseteq V$ is *stable* if and only if $E{:}X = \varnothing$, $\kappa : V \to \{-k, \ldots, 0, k\}$. If $X = \kappa^{-1}(0)$, then $\Sigma{:}(V \setminus X)$ is zero free so it has $\chi^*_{\Sigma \setminus X}(\lambda - 1)$ proper colorations.

Theorem 64. $\chi_\Sigma(\lambda) \;=\; \sum\limits_{\substack{X \subseteq V \\ X \text{ is stable}}} \chi^*_{\Sigma \setminus X}(\lambda - 1).$

Notes for 10 Mar. 2017 – Amelia Mattern.

We continue with the proof of Theorem 64 from last class.

Proof. Let Σ be a signed graph. Fix an odd positive integer λ and let

$$\mathrm{K}_\Sigma := \mathrm{K}_\Sigma(\lambda) := \{\kappa \mid \kappa \text{ is a proper } \lambda\text{-coloration of } \Sigma\}.$$

Also let

$$\mathscr{B} := \{(X, \kappa^*) \mid \kappa^* \text{ is a proper } \lambda\text{-coloration of } \Sigma \setminus X \text{ and } X \text{ is stable}\}.$$

Let $\kappa \in \mathrm{K}_\Sigma$ and let $X = \kappa^{-1}(0)$. Then X is stable, since otherwise there is an improper edge in $E{:}X$. Also, $\Sigma \setminus X$ is properly colored without 0. This gives a mapping $\gamma : \mathrm{K}_\Sigma \to \mathscr{B} : \kappa \mapsto (X, \kappa|_{V \setminus X})$.

Conversely, let $(X, \kappa^*) \in \mathscr{B}$. Then define $\kappa : V \to \mathbf{C}$ (where $\mathbf{C}$ is the color set) by

$$\kappa(v) = \begin{cases} 0 & \text{if } v \in X, \\ \kappa^*(v) & \text{if } v \notin X. \end{cases}$$

Let e be an edge of Σ. Then $e \notin E : X$. If $e \in E{:}(V \setminus X)$ it is κ^*-colored and therefore proper. If e is an edge between X and $V \setminus X$, then one endpoint is colored 0 and the other is colored non-zero, and thus it is proper. If e is a loose edge, then it is in $\Sigma \setminus X$ for every X. But since κ^* is a proper coloration of $\Sigma \setminus X$, there can be no loose edges if there exists even one X. Luckily, we are saved by the null set as $\varnothing$ is always stable. Therefore, κ is a proper coloration of Σ. So we now have mappings $\gamma : \mathrm{K}_\Sigma \to \mathscr{B}$ and $\delta : \mathscr{B} \to \mathrm{K}_\Sigma$. It's too easy for words to check that $\gamma\delta$ and $\delta\gamma$ are the appropriate identity functions. Therefore γ and δ are bijections.

That was for any odd positive integer λ. Now let λ be anything and use the fact that, since we have a polynomial equation for all positive odd integers, the polynomial function extends to any argument. Thus,

$$\chi_\Sigma(\lambda) = \sum_{\substack{X \subseteq V \\ X \text{ stable}}} \chi^*_{\Sigma \setminus X}(\lambda - 1). \quad \square$$

Good examples.

Now we look at some good special examples of signed graphs.

Definition 23. A signed graph is *full* if every vertex supports an unbalanced edge. $\Sigma^\bullet$ denotes Σ with a half edge or negative loop added to every vertex that did not already have one.

Example 3. For a graph Γ, $(\pm\Gamma)^\bullet$ (written more simply as $\pm\Gamma^\bullet$) is the full signed expansion of Γ. Here Γ is a link graph.

Corollary 65. $\chi_{\Sigma^\bullet}(\lambda) = \chi^*_\Sigma(\lambda - 1)$.

Proof. X stable implies $X = \varnothing$. $\square$

Example 4. For a link graph Γ,

$$\chi_{\pm\Gamma^\bullet}(\lambda) = \chi^*_{\pm\Gamma}(\lambda - 1).$$

In particular,

$$\chi_{\pm K_n^\bullet}(\lambda) = \chi^*_{\pm K_n}(\lambda - 1) = 2^n \left(\frac{\lambda - 1}{2}\right)_n.$$

Example 5. For a link graph Γ,

$$\chi_{-\Gamma^\bullet}(\lambda) = \chi^*_{-\Gamma}(\lambda - 1).$$

In particular,

$$\chi_{-K_n^\bullet}(\lambda) = \chi^*_{-K_n}(\lambda - 1).$$

Example 6. Let Σ be $\pm K_n$ with half edges at all vertices $v \in W$ for some $W \subseteq V$. Then

$$\chi_\Sigma(\lambda) = \chi^*_\Sigma(\lambda - 1) + \sum_{v \notin W} \chi^*_{\Sigma\setminus v}(\lambda - 1) = \chi^*_{\pm K_n}(\lambda - 1) + (n - \#W)\chi^*_{\pm K_{n-1}}(\lambda - 1).$$

Let's compute $\chi^*_{\pm K_n}(\mu)$. Let $\mu = 2k$ and color using $\pm[k]$. The only restriction on a proper coloration κ is that $|\kappa(v_j)| \neq |\kappa(v_i)|$ for $j \neq i$. So κ is proper if and only if $|\kappa|$ is proper on K_n. It follows that $\chi^*_{\pm K_n}(2k) = 2^n \chi_{K_n}(k)$. Thus $\chi^*_{\pm\Gamma}(2k) = 2^n\chi_\Gamma(k)$. This implies that

$$\chi_{\Gamma^\bullet}(\lambda) = \chi^*_{\pm\Gamma}(\lambda - 1) = 2^n\chi_\Gamma\Big(\frac{\lambda - 1}{2}\Big).$$

The lattice $\mathrm{Lat}(\pm K_n^\bullet)$ is the *Dowling lattice* of rank n of the group $\{+,-\}$. Every group $\mathfrak{G}$ has a Dowling lattice of rank n for all $n > 0$. The customary notation for this lattice is $Q_n(\mathfrak{G})$. The Dowling lattice is an important example in matroid theory. A paper by Kahn and Kung [9] demonstrates that it has a role analogous to that of projective spaces, although not as central.

As for the hyperplane arrangement, we have $\mathscr{H}[\Sigma^\bullet] = \mathscr{H}[\Sigma] \cup \{\text{all coordinate hyperplanes}\}$. This is why full graphs are useful: the coordinate hyperplanes provide a kind of frame that simplifies many calcuations. For example,

$$\mathscr{H}[\pm K_n] = \text{ the root system arrangement } \mathscr{D}_n$$

while

$$\mathscr{H}[\pm K_n^\bullet] = \text{ the root system arrangement } \mathscr{B}_n = \mathscr{C}_n.$$

Notes for 13 Mar. 2017 – Josh Carey.

8.2. **Chromatic Number.**

The *chromatic number* of a signed graph Σ is

$$\chi(\Sigma) = \min\{\lambda \geq 0 \mid \chi_\Sigma(\lambda) > 0\}.$$

Here $\chi_\Sigma(\lambda)$ denotes the chromatic function, i.e., the number of proper λ-colorations. This definition is unambiguous if Σ has a proper coloration at all. In the exceptional case that Σ has a loose edge or positive loop, hence no proper colorations, we define $\chi(\Sigma) = \infty$. The chromatic number is always positive (counting ∞ as positive) with one weird exception: When $\Sigma = \varnothing$, then $\chi(\Sigma) = 0$ since there is one proper coloration with no colors and there are no colorations at all with a positive number of colors.

Proposition 66. *Assume $\Sigma \neq \varnothing$. For a nonnegative integer λ, $\chi_\Sigma(\lambda) > 0$ if and only if $\lambda \geq \chi(\Sigma)$.*

Proof. To avoid triviality we assume Σ has no positive loops or loose edges. We want to prove that if $\chi_\Sigma(\lambda) > 0$ and $\lambda' > \lambda$, then $\chi_\Sigma(\lambda') > 0$. In fact, we prove that χ_Σ is strictly increasing for $\lambda \geq \chi_\Sigma$.

Assume Σ properly colored with λ colors. There are two cases depending on parity.

If λ is even, we colored with color set $\pm\left[\frac{\lambda}{2}\right]$, omitting 0. If we enlarge the color set to $\left\{-\frac{\lambda}{2}, \ldots, 0, \ldots, \frac{\lambda}{2}\right\}$, we still have the same coloration as well as the possibility of recoloring any one vertex with 0, so $\chi_\Sigma(\lambda+1) > \chi_\Sigma(\lambda) > 0$.

If λ is odd, we colored with $\kappa : V \to \{-\frac{\lambda-1}{2}, \ldots, 0, \ldots, \frac{\lambda-1}{2}\}$. We would like to replace the color 0 by the colors $\pm\frac{\lambda+1}{2}$. We can do this because $\kappa^{-1}(0)$ is stable, so we can recolor $\kappa^{-1}(0)$ by any combination of 0 and $\pm\frac{\lambda+1}{2}$. If 0 was used in κ, that gives us at least two proper colorations derived from κ. If not, we can change the color of any one vertex to $\pm\frac{\lambda-1}{2}$, thus giving at least 2 additional derived proper colorations. Therefore $\chi_\Sigma(\lambda+1) > \chi_\Sigma(\lambda) > 0$. $\square$

(We made no effort to prove the strongest kind of increase. We know that asymptotically $\chi_\Sigma(\lambda) \sim \lambda^n$ because that is the leading term of the chromatic quasipolynomial, as we show next.)

There will be more to say about the chromatic number later, when we prove the signed-graph analog of the famous (or notorious) Brooks' Theorem for unsigned graphs.

8.3. **Chromatic Coefficients.**

Now we investigate the coefficients of the chromatic polynomials. Here we separate the total chromatic polynomial (odd arguments) from the balanced chromatic polynomial (even arguments). I use a trick of notation to combine the two cases, as their proofs are virtually identical. The notations in square brackets, [b] and [balanced], apply to the zero-free chromatic polynomial, χ^*_Σ (or χ^{b}_Σ), that is, even values of λ; while omitting the bracketed notations gives the formulas for the total chromatic polynomial, $\chi_\Sigma(\lambda)$, with odd values of λ.

Theorem 67. *Let Σ be a signed graph with no positive loops or loose edges. Then $\chi^{[\mathrm{b}]}_\Sigma(\lambda) = w^{[\mathrm{b}]}_0\lambda^n + w^{[\mathrm{b}]}_1\lambda^{n-1} + \cdots + w^{[\mathrm{b}]}_n\lambda^0$, where each $w^{[\mathrm{b}]}_i$ is a constant relative to λ, but depends on Σ. Then $w^{[\mathrm{b}]}_0 = 1$, $(-1)^i w^{[\mathrm{b}]}_i > 0$ for $i \leq n - i^{[\mathrm{b}]}_0$ and $w^{[\mathrm{b}]}_i = 0$ for $i > n - i^{[\mathrm{b}]}_0$, where $i_0 = n - b(\Sigma)$ and $i^{\mathrm{b}}_0 = n - c(\Sigma)$.*

Proof. The proof is in two parts. The first depends on the subset expansions to show that both $\chi^{[\mathrm{b}]}_\Sigma(\lambda)$ are polynomials. The second applies induction on the number of edges to get the coefficient signs.

In the first step of the first part we demonstrate that $\chi^{[\mathrm{b}]}_\Sigma(\lambda)$ has degree n. From Theorem **??**,

$$\chi^{[\mathrm{b}]}_\Sigma(\lambda) = \sum_{\substack{S \subseteq E \\ [\text{balanced}]}} (-1)^{\#S} \lambda^{b(\Sigma|S)}.$$

The largest $b(\Sigma|S)$ can be is n and that only when S contains no links and no unbalanced edges. So $S = \varnothing$ and the coefficient of λ^n is 1. Therefore, $\chi^{[\mathrm{b}]}_\Sigma(\lambda)$ is indeed monic of degree n.

The lowest value of $b(\Sigma|S)$ is attained when $S = E$, in which case $b(\Sigma|S) = b(\Sigma)$; therefore $i_0 = n - b(\Sigma)$.

Considering $\chi^{\mathrm{b}}_\Sigma(\lambda)$, the lowest value of $b(\Sigma|S)$ over balanced sets S is $c(\Sigma)$. This can be seen from the following computation for a balanced edge set: $b(\Sigma|S) = c(\Sigma|S)$ (because S

is balanced) $\geq c(\Sigma)$ (because $S \subseteq E$). Therefore $b(\Sigma|S) \geq c(\Sigma)$. If S is a maximal forest in Σ, then $c(S) = c(\Sigma)$; but since S is balanced, we also have $b(S) = c(S)$. Therefore $b(\Sigma|S) = c(\Sigma)$ is attained. So, $i_0^{\mathrm{b}} = n - c(\Sigma)$.

There is one gap here. We have not shown that $w_i^{[\mathrm{b}]} \neq 0$ for $0 \leq i \leq n - i_0^{[\mathrm{b}]}$. That will be proved in the second half of the proof.

We now proceed to prove the coefficient signs by induction on $\#E$. We begin with the case where $E = \varnothing$. Then $\chi_\Sigma^{[\mathrm{b}]}(\lambda) = \lambda^n$. We know (Theorem **??**) that for any link e and (for the total polynomial) any unbalanced edge e, we have

$$\chi_\Sigma^{[\mathrm{b}]}(\lambda) = \chi_{\Sigma\setminus e}^{[\mathrm{b}]}(\lambda) - \chi_{\Sigma/e}^{[\mathrm{b}]}(\lambda) = \sum_{i=0}^{n} w_i^{[\mathrm{b}]}(\Sigma \setminus e)\lambda^{n-i} - \sum_{i=0}^{n-1} w_i^{[\mathrm{b}]}(\Sigma/e)\lambda^{n-1-i},$$

because $|V(\Sigma/e)| = n - 1$. Changing indices of summation, our expression becomes

$$\begin{aligned} \sum_{j=0}^{n} w_j^{[\mathrm{b}]}(\Sigma)\lambda^{n-j} &= \sum_{j=0}^{n} w_j^{[\mathrm{b}]}(\Sigma \setminus e)\lambda^{n-j} + \sum_{j=i}^{n}(-w_{j-1}^{[\mathrm{b}]}(\Sigma/e))\lambda^{n-j} \\ &= w_0^{[\mathrm{b}]}(\Sigma \setminus e)\lambda^n + \sum_{j=1}^{n} \left[w_j^{[\mathrm{b}]}(\Sigma \setminus e) - w_{j-1}^{[\mathrm{b}]}(\Sigma/e)\right]\lambda^{n-j}. \end{aligned}$$

By the induction hypothesis

$$\begin{aligned} w_j^{[\mathrm{b}]}(\Sigma) &= w_j^{[\mathrm{b}]}(\Sigma \setminus e) - w_{j-1}^{[\mathrm{b}]}(\Sigma/e) \\ &= (-1)^j|w_j^{[\mathrm{b}]}(\Sigma \setminus e)| - (-1)^{j-1}|w_{j-1}^{[\mathrm{b}]}(\Sigma/e)| \\ &= (-1)^j\big(|w_j^{[\mathrm{b}]}(\Sigma \setminus e)| + |w_{j-1}^{[\mathrm{b}]}(\Sigma/e)|\big). \end{aligned}$$

This proves that $w_j^{[\mathrm{b}]}(\Sigma) \neq 0$ if either $w_j^{[\mathrm{b}]}(\Sigma \setminus e)$ or $w_{j-1}^{[\mathrm{b}]}(\Sigma/e)$ is nonzero, and that its sign is $(-1)^j$.

We still have to prove that at least one of $w_j^{[\mathrm{b}]}(\Sigma \setminus e)$ and $w_{j-1}^{[\mathrm{b}]}(\Sigma/e)$ is nonzero. [**MORE PROOF NEEDED.**]

□

Notes for 15 Mar. 2017 – (Frosty the Snowman).

Snow day!

Notes for 17 Mar. 2017 – Micah Loverro.

8.4. **Chromatic Number Again.**

A *coloration* of a signed graph Σ is a function $\kappa : V \to \mathbb{Z}$. (We could reasonably replace $\mathbb{Z}$ with other color sets but they have to be sign-symmetric, as you will see.)

Definition 24. A *coloration* of Σ is a function $\kappa : V \to \mathbb{Z}$. It is *zero free* if $0 \notin \operatorname{Im} \kappa$. It is a $2k+1$-*coloration* if $\operatorname{Im} \kappa \subseteq [-k, k]$. It is a $2k$-*coloration* if $\operatorname{Im} \kappa \subseteq \pm[k]$.

A coloration κ of Σ is proper if

1. $\kappa(v) \neq \sigma(e)\kappa(w)$ whenever $e{:}vw$ is a link or a loop,
2. $\kappa(v) \neq 0$ whenever $e{:}v$ is a half edge, and
3. Σ has no loose edges.

The *chromatic number* of Σ is

$$\chi(\Sigma) = \min\{\lambda \geq 0 \mid \text{ there is a proper } \lambda\text{-coloration of } \sigma\}.$$

We also define the *chromatic function*, $\chi_\Sigma(\lambda) :=$ the number of proper λ-colorations of Σ, for integers $\lambda \geq 0$.

Here are two theorems about the nature of the chromatic function.

Theorem 68. *If $\lambda \geq \chi(\Sigma)$, there is a proper λ-coloration of Σ. In fact, $\chi_\Sigma(\lambda)$ is increasing.*

Theorem 69. *$\chi_\Sigma(\lambda)$ restricted to the even integers is a monic polynomial. So is $\chi_\Sigma(\lambda)$ restricted to the odd integers.*

That is, $\chi_\Sigma(\lambda)$ is almost a polynomial; it is a quasipolynomial of period 1 or 2. For odd integers $\lambda = 2k+1$, we call $\chi_\Sigma(\lambda) = \chi_\Sigma(2k+1)$ the chromatic polynomial, and for even integers $\lambda = 2k$ we call $\chi_\Sigma(2k) = \chi^*_\Sigma(\lambda) = \chi^b_\Sigma(\lambda)$ the zero-free or balanced chromatic polynomial. [**WAS quasipolynomial ALREADY DEFINED?**]

Switching a Coloration.

For a switching function $\zeta : V \to \{+, -\}$, define the switched coloration $\kappa^\zeta(v) := \kappa(v)\zeta(v)$. Then:

Lemma 70. *A coloration κ is a proper coloration on Σ if and only if κ^ζ is*

There is the following easy upper bound for graphs. $\Delta(\Gamma)$ is the maximum degree of a vertex in Γ.

Theorem 71. *If Γ is a simple graph, then $\chi(\Gamma) \leq \Delta(\Gamma) + 1$.*

This bound can be improved in many ways, and has been. For instance, one can prove from Euler's polyhedral formula that every planar simple graph has a vertex with degree at most 5. This can be used to prove the

Theorem 72 (6-Color Theorem)**.** *Every planar graph is 6-colorable.*

Proof. The proof is by induction on $\#V$. The lemma allows us to find a vertex v of degree less than or equal to 5, then $\Gamma \setminus \{v\}$ is 6-colorable by induction. Then we just choose a color for v, there being at most 5 colors to avoid. $\square$

Given any class of graphs satisfying a similar lemma, one could infer a chromatic number bound by a similar proof. There are a number of examples in the literature. One almost trivial example is the preceding easy theorem, that $\chi(\Gamma) \leq \Delta(\Gamma)+1$. A famous improvement is known as Brooks' theorem. It may look elementary, since it reduces the trivial bound only by 1, but the proof is not simple.

Theorem 73 (Brooks' Theorem (1941))**.** *For a simple graph* Γ*,* $\chi(\Gamma) \leq \Delta(\Gamma)$ *unless* $\Gamma = K_n$ *or an odd circle.*

I only say "simple graph" because that's how the theorem is usually stated. Multiple edges can only increase the upper bound $\Delta(\Gamma)$; they don't change the chromatic number. Loops do violate the bound—but a loop is an odd circle!

Recently there has appeared an analogous theorem for signed simple graphs by Máčajová, Raspaud, and Škoviera [10].

Theorem 74 (Brooks' Theorem for Signed Simple Graphs)**.** *Let* Σ *be a signed simple graph. Then* $\chi(\Sigma) \leq \Delta(\Sigma)$ *except when* $[\Sigma] = [+K_n]$*,* C_{2n+1}^{+}*, or* $[C_{2n}^{-}]$*.*

Here C_n^{ε} means a circle C_n with sign $\sigma(C_n) = \varepsilon$.

A *simply signed graph* is a signed graph with no loose edges or positive loops, at most one unbalanced edge at each vertex, and no parallel links of the same sign. That is not the same as a signed simple graph; the latter has no loops and no parallel edges of any sign. Theorem 74 has been strengthened and the proof made elegant in a follow-up paper. (The next lecture!)

Notes for 20 Mar. 2017 – (?)

Notes for 22 Mar. 2017 – (Josh Carey).

8.5. **Brooks' Theorem for Signed Simple Graphs.**

[Brooks' Theorem for signed graphs: [10] simple; [5] all, and list coloring.]

Notes for 24 Mar. 2017 – Amelia Mattern.

We continue with the proof of Brooks' Theorem for signed simple graphs.

Case 2: If there are no "backwards" edges, then there are no edges from T_1 to any vertices not in T_1; otherwise this contradicts our choice of u. Let w be the child of u_1 in T_1. Then $\deg_{\Sigma_1}(u_1) \geq 2$ implies there exists an edge from u_1 to some vertex $x \neq w$ in T_1. The existence of the edge u_1x implies that $\Sigma_1 \setminus \{w\}$ is connected. Also, $v = u_1$ has degree less than or equal to $\Delta - 1$ in Σ_1. So by lemma 3 there exists a Δ-coloration of Σ_1 such that $\kappa(u_1) \neq 0$. So we can choose $\kappa(u_1) = c'$. We can prove a similar fact for u_2 in Σ_2. In conclusion, we can color Σ_1 and Σ_2 so $\kappa(u_1) = \kappa(u_2) = c' \neq 0$.

In $\Sigma' := \Sigma \setminus (V_1 \cup V_2)$, $\Delta(\Sigma') \leq \Delta$ and $\deg_{\Sigma'}(u) \leq \Delta - 2$. So by lemma 2, there exists κ', a coloring of Σ' with $\kappa'(u) \neq 0$. Assume we switched Σ so uu_1 and uu_2 are negative. Then we can choose κ' so that $\kappa'(u) = c'$. Note that $\Sigma \setminus (V_1 \cup V_2)$ is connected. So there exists a connected ordering of V' with u as the last vertex. We continue the coloring from Σ_1 and Σ_2 into Σ' along this ordering by greedily coloring up to u. Then $\deg_{\Sigma}(u) = \Delta$, but we know u has two neighbors of the same color. Thus u has a color available and the coloring is proper. $\square$

Remarkably, a vast generalization was proved quite quickly after the original signed Brooks' theorem appeared.

Definition 25. We call a finite-set-valued vertex function $L : V \to \mathscr{P}_{\text{fin}}(\mathbb{Z})$ a *list function* and the set $L(v)$ is the *list* for vertex v. Given L, if there exists a proper coloration κ of Γ such that $\kappa(v) \in L(v)$ for all $v \in V$, we say Γ is *L-colorable.*

Let $f : V \to \mathbb{Z}_{\geq 0}$. If Γ is L-colorable for every list function L such that $L(v) = f(v)$ for all $v \in V$, then we say Γ is *f-list-colorable.*

If Γ is f-list-colorable for the constant function $f \equiv k \in \mathbb{Z}_{\geq 0}$, then Γ is called *k-list-colorable* or *k-choosable.*

The *list chromatic number* of Γ is $\chi_{\text{list}}(\Gamma) := \min\{k \mid \Gamma \text{ is } k\text{-list-colorable}\}$.

The list analog of Brooks' Theorem is a characterization of Δ-list-colorable simple graphs. The generalization to signed graphs is:

Theorem 75 (Fleiner and Wiener [5])**.** *A connected signed graph Σ in which all edges are links, having no multiple edges of the same sign, is $\Delta(\Sigma)$-list-colorable unless Σ is a balanced K_n, a positive odd circle, a negative even circle, $\pm K_n$ for any n, or $\pm C_n$ for odd $n \geq 3$.*

Notes for 27 Mar. 2017 – Micah Loverro.

8.6. **Colorations and Hyperplanes: The Geometry of λ-Coloring.**

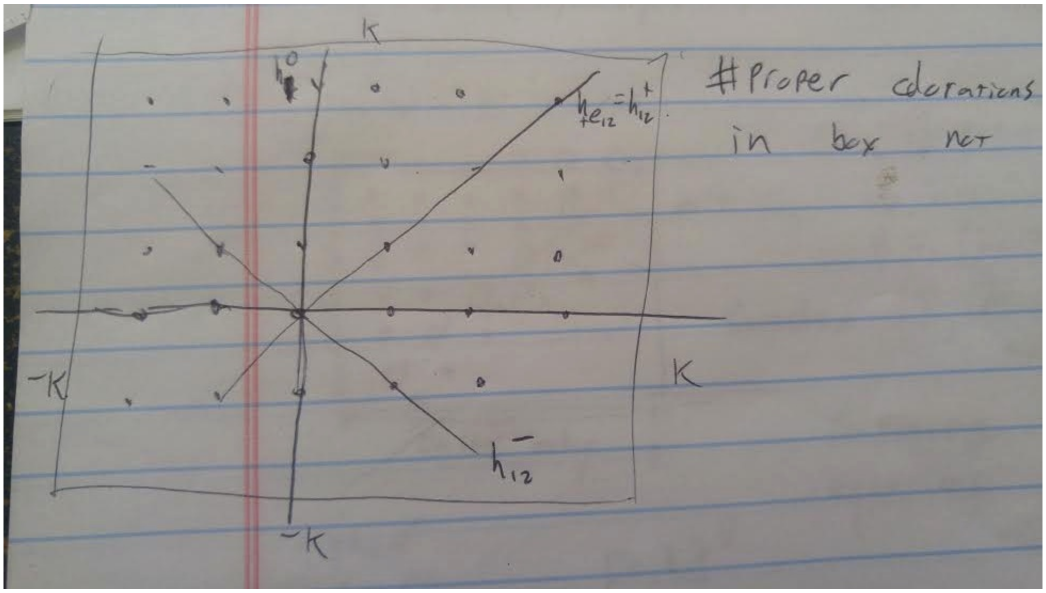
Throughout, we assume Σ has no loose edges. Loose edges trivialize everything!

Let κ be a coloration. We view now κ as an element of $([-k,k] \cap \mathbb{Z})^V \subseteq \mathbb{R}^V$. We regard the interval $[-k,k]^V = k[-1,1]^V$ as a rescaling of the centrally symmetric hypercube $[-1,1]^V$, which makes k into a dilation (or expansion) factor.

Integral Ehrhart theory.

For the time being, however, let's discuss unsigned graph coloring, which is simpler. A coloration will be a function $V \to \{0, 1, \ldots, \lambda - 1\}$. Then we use the polytope $(\lambda-1)[0,1]^V$, where $\lambda - 1$ is the dilation factor for the unit hypercube, so $\kappa \in (\lambda-1)[0,1]^V \cap \mathbb{Z}^V$. The condition for κ to be proper is that (i) $\kappa(v) \neq \kappa(u)\sigma(e)$ for every link $e{:}uv \in E$, i.e., $\kappa \notin h_e$ where h_e is the hyperplane $x_u = x_v$ corresponding to e, and (ii) $\kappa(v) \neq 0$ for each half edge $e{:}v \in E$, i.e., $\kappa \notin h_e{:}x_v = 0$. Thus, finding proper colorations is the same as finding integral points in a box which is the dilated unit hypercube and that avoid all the forbidden edge hyperplanes h_e. The number of proper λ-colorations is the number of lattice points in $\mathbb{Z}^V$ that are in the box but are not in any hyperplane. The number of such points is given by the Ehrhart theory of inside-out polytopes.

[picture 1]



Ehrhart Theory. [An introductory reference for Ehrhart theory is Beck and Robins [1].]

Let $P \subseteq \mathbb{R}^n$ be a closed integral polytope whose dimension is d (we say "integral d-polytope" for short); that means its vertices are points of the integer lattice (which is the only lattice we deal with in Ehrhart theory) and that the affine subspace spanned by P is d-dimensional. For an integer $t > 0$, define the Ehrhart function $E_P(t) :=$ the number of lattice points in the dilated polytope tP.

Theorem 76 (Ehrhart Polynomial)**.** *If P is an integral d-polytope, then $E_P(t)$ is a polynomial in t of degree d and with leading coefficient equal to the volume of P.*

This theorem applies to nonconvex as well as convex polytopes, but from now on we assume all polytopes are convex, as that is all (actually, more than) we need for graph coloring.

Let P° denote the interior of P. Then $E_{P^\circ}(t) =$ the number of lattice points in the interior of tP.

Theorem 77 (Ehrhart Reciprocity)**.** *For an integral d-polytope P, $E_{P^\circ}(t) = (-1)^d E_P(-t)$.*

For proper colorations we are interested in the function $E_P(t)$ or $E_{P^\circ}(t)$ where P is the unit hypercube $Q^n := [0,1]^n$ in $\mathbb{R}^n$ and $t = \lambda - 1$ in the former case, $t = \lambda + 1$ in the latter (see below). I explained how to use the closed hypercube; for proper colorations, however, it is better to use the open hypercube. If we use the color set $[\lambda] = \{1, \ldots, \lambda\}$, then the number of lattice points in P° where $P = (\lambda+1)[0,1]^V$ is the total number of colorations of Γ. Another way to look at this is to divide the lattice points and maintain the same polytope; then the number of colorations is the number of $\frac{1}{t}$-lattice points in the unit hypercube.

For proper colorations we forbid points in an edge hyperplane. The general setup is that of a pair $(P, \mathscr{H})$ consisting of a convex polytope and an arrangement of hyperplanes, called an *inside-out polytope*. We have a dilation factor t, a positive integer, and we form the Ehrhart function $E_{P,\mathscr{H}}(t) :=$ the number of $\frac{1}{t}$-fractional points in P that are not in any of the hyperplanes of $\mathscr{H}$; that is,

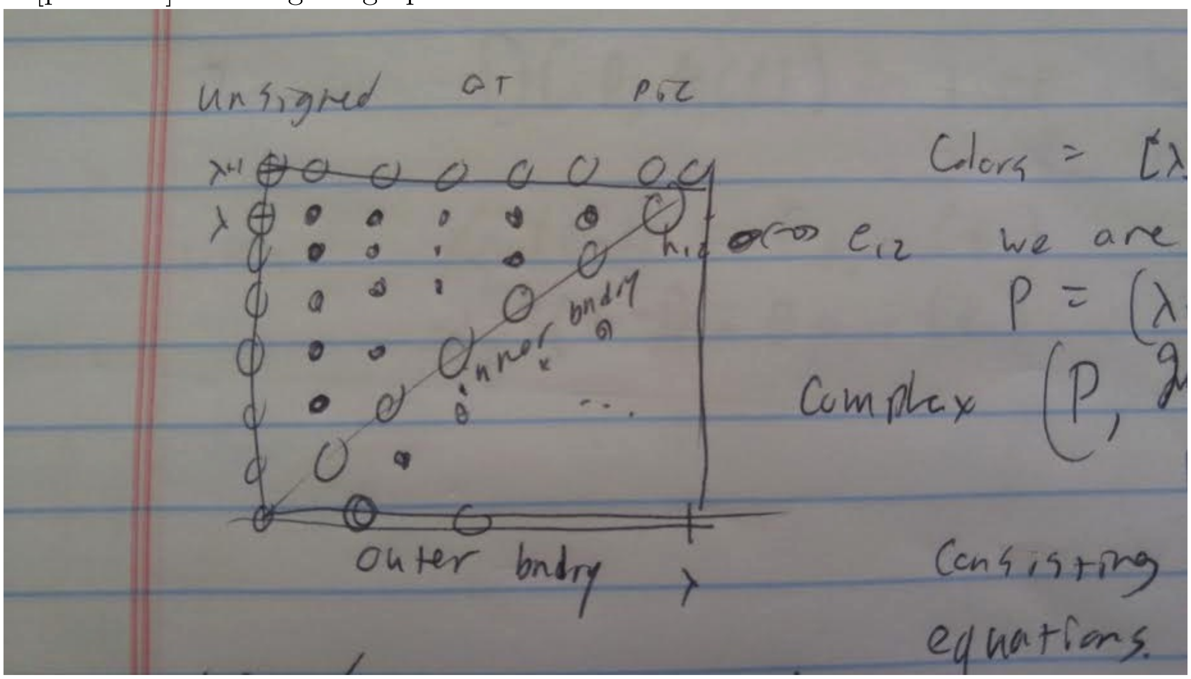
$$E_{P,\mathscr{H}}(t) := \#\Big(P \cap \frac{1}{t}\mathbb{Z}^n \setminus \bigcap \mathscr{H}\Big)$$

(I'm now writing $n = \#V$ instead of V). Similarly, the open Ehrhart function is

$$E^{\circ}_{P,\mathscr{H}}(t) := \#\Big(P^{\circ} \cap \frac{1}{t}\mathbb{Z}^n \setminus \bigcap \mathscr{H}\Big).$$

The advantage of using a fractional lattice instead of dilating the polytope is that the hyperplanes need not be self-similar under dilation—a property that belongs only to homogeneous hyperplanes. (In truth, we only use homogeneous hyperplanes for graph coloring; but I think it is easier to see the whole picture when we keep the polytope and hyperplanes fixed.)

[picture 2] for unsigned graphs



The fractional lattice still has an integral polytope. That suffices for unsigned graphs, but for signed graphs (this is not obvious but you will see why) we need the extension to rational polytopes. That means P has vertices with rational coordinates that do not have to be integers.

Rational Ehrhart theory.

This is the Ehrhart theory for polytopes with vertices in $\mathbb{Q}^d$. The definitions of the closed and open Ehrhart functions are exactly the same as before, but the theorems are not. We need a new invariant of P: its *denominator* $D(P)$ is the least common denominator of all coordinates of vertices of P; equivalently, it is the smallest number such that the vertex set of P is contained in the $D(P)$-fractional lattice $\frac{1}{D(P)}\mathbb{Z}^n$; also equivalently, it is the least positive integer t such that all vertices of tP are integral.

To handle rational polytopes we need a new concept. A function of positive integers, $f : \mathbb{Z}_{>0} \to \mathbb{R}$, is called a *quasipolynomial* if it is given by a cyclically repeating series of polynomials. That means there are, say, π polynomials $f_0, f_1, \dots, f_{p-1}$ such that $f(n) = f_{n \mod \pi}(n)$, where $n \mod \pi$ is the least nonnegative residue of n modulo π. The smallest π for which that is true is called the *period* of f. Quasipolynomials appear in number theory and analysis and have some equivalent forms that appear quite different at first sight, but our interest comes from geometry through the first fundamental theorem of rational Ehrhart theory:

Theorem 78 (Ehrhart Quasipolynomial)**.** *Let P be a rational d-polytope in $\mathbb{R}^d$. Then $E_P(t)$ and $E^\circ_P(t)$ are quasipolynomials with degree d and leading coefficient vol$P$. The period of each quasipolynomial divides $D(P)$.*

Thus we have polynomials $E_{P,0}(t), \dots, E_{P,\pi-1}(t), E_{P,\pi}(t) = E_{P,0}(t)$ with $\pi | D(P)$ and $E_P(t) = E_{P,t \mod \pi}(t)$.

Given a quasipolynomial function of positive integers, the formula $f(n) = f_{n \mod \pi}(n)$ automatically extends it to all integers, in particular to negative integers. That leads up to the second fundamental theorem of rational Ehrhart theory:

Theorem 79 (Ehrhart Reciprocity)**.** *For a rational d-polytope in $\mathbb{R}^d$, $E^\circ_P(t) = (-1)^d E_P(-t)$.*

This implies, in particular, that the closed and open Ehrhart quasipolynomials have the same period.

In coloring signed graphs we only have to deal with at most two polynomials.

Theorem 80. *The denominator $D(P, \mathscr{H}[\Sigma]) = 1$ if Σ is balanced, and it $= 2$ if Σ is not balanced. Moreover,*

$$\chi_\Sigma(t-1) = E_{P,0}(t) \text{ and } \chi^b_\Sigma(t-1) = E_{P,1}(t).$$

I will prove this shortly through geometry supplemented with a little algebra.

Notes for 29 Mar. 2017 – Ted Ofner.

8.7. **The Ehrhart Setup for Signed-Graph Coloring.**

Let Σ be a signed graph and $\mathscr{H}[\Sigma]$ its corresponding homogeneous arrangement of hyperplanes. Given $\lambda \in \mathbb{Z}_{>0}$, let $\chi_\Sigma(\lambda)$ be the number of proper λ-colorations of Σ. We have seen that,

$$\chi_\Sigma(\lambda) = \begin{cases} \left|(-\frac{\lambda+1}{2}, \frac{\lambda+1}{2})^n \cap \mathbb{Z}^n \setminus \bigcup \mathscr{H}[\Sigma]\right| & \text{for odd } \lambda, \\ \left|[(-\frac{\lambda+1}{2}, \frac{\lambda+1}{2}) \setminus 0]^n \cap \mathbb{Z}^n \setminus \bigcup \mathscr{H}[\Sigma]\right| & \text{for even } \lambda. \end{cases}$$

We can see the basic ideas of Ehrhart polynomials, but the setup is not quite correct. Firstly, we do not have a single polytope we can consider. To that end, we adjust our color scheme.

Recall that for odd $\lambda = 2k+1$ we use colors

$$\{-k, -(k-1), \dots, -1, 0, 1, \dots, k-1, k\},$$

Consider the subsitution $k = \frac{\lambda-1}{2}$. After this, our odd color set becomes

$$\Big\{ -\frac{\lambda-1}{2}, -\frac{\lambda-3}{2}, \dots, -1, 0, 1, \dots, \frac{\lambda-3}{2}, \frac{\lambda-1}{2} \Big\},$$

so it is clear that we count vectors in $(-\frac{\lambda+1}{2}, \frac{\lambda+1}{2})^n \cap \mathbb{Z}^n$.

However, for even $\lambda = 2k$, we use colors

$$\{-k, -(k-1), \dots, -1, 1, \dots, k-1, k\}.$$

Our analogous substitution to the odd case is $k = \frac{\lambda}{2}$, which gives us a color set,

$$\Big\{ -\frac{\lambda}{2}, -\frac{\lambda-2}{2}, \dots, -1, 1, \dots, \frac{\lambda-2}{2}, \frac{\lambda}{2} \Big\}.$$

This set is the same size as

$$\Big\{-\frac{\lambda-2}{2},-\frac{\lambda-4}{2},\dots,-1,0,1,\dots,\frac{\lambda-2}{2},\frac{\lambda}{2}\Big\}.$$

Then shifting by $-\frac{1}{2}$ gives a color set

$$\Big\{-\frac{\lambda-1}{2},-\frac{\lambda-3}{2},\dots,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\dots,\frac{\lambda-3}{2},\frac{\lambda-1}{2}\Big\}.$$

Thus, in both the odd and even case we can count vectors away from $\mathscr{H}[\Sigma]$ with coordinates in $\{\frac{\lambda-1}{2}-i, i=0,1,\dots,\lambda-1\}$. The polytope we initially consider, then, is the open polytope $(\lambda+1)(-\frac{1}{2},\frac{1}{2})^n$, and we can write

$$\chi_\Sigma(t-1)=\Big|\Big(-\frac{1}{2},\frac{1}{2}\Big)^n\cap\frac{1}{t}\Big(\mathbb{Z}+\big(\frac{t}{2}\text{ mod }1\big)\Big)^n\setminus\bigcup\mathscr{H}[\Sigma]\Big|$$

We want to use a polytope with unit volume so our resultant Ehrhart quasipolynomial is monic, which will make it exactly the chromatic quasipolynomial. The current setup, however, does not quite get us all the way, since there is the problem of the dependence on t of the scaled lattice $\mathbb{Z}+(\frac{t}{2}$ mod 1. To fix this, we adjust our color sets one more time. Instead of considering $\{\frac{t-2}{2}-i, i=0,1,\dots,t-2\}$ in $(-\frac{t}{2},\frac{t}{2})$, we shift both sets by $\frac{t}{2}$ so we consider $\{0,1,\dots,t-1\}$ in $(0,t)$. This means we can always consider the lattice $\mathbb{Z}$, but our polytope is no longer centered; namely, $(-\frac{1}{2},\frac{1}{2})^n$ is shifted by the vector $\frac{1}{2}\mathbf{1}=(\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2})$ ($\mathbf{1}$ is the all-ones vector) to become $(0,1)^n$. The theory of Ehrhart polynomials has no problem with this, as long as we shift our hyperplane arrangement correspondingly.

Let, therefore, $\hat{\mathscr{H}}[\Sigma]=\mathscr{H}[\Sigma]+\frac{1}{2}\mathbf{1}$, i.e., we shift all hyperplanes by the vector $\frac{1}{2}\mathbf{1}$.

This gives us our desired Ehrhart description of χ_Σ.

Theorem 81. $\chi_\Sigma(t-1)=|(0,1)^n\cap\frac{1}{t}\mathbb{Z}^n\setminus\hat{\mathscr{H}}[\Sigma]|$. □

8.7.1. *The Denominator of the Chromatic Quasipolynomial.*

If Σ is balanced, switch it to be all positive. Then all hyperplanes in $\mathscr{H}[\Sigma]$ have the form $x_v=x_u$. This equation is unchanged by a shift of $(\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2})$, so all hyperplanes in $\hat{\mathscr{H}}[\Sigma]$ have the same form as before. The inside-out polytope $([0,1]^n,\hat{\mathscr{H}}[\Sigma])$ has vertices determined only by equations of the form $x_v=x_u$, $x_v=0$, or $x_v=1$. Combining such equations can only determine a point with coordinates all 0 or 1. This the denominator of χ_Σ is 1 when Σ is balanced.

If Σ is unbalanced, it may have some negative edges. In terms of $\mathscr{H}[\Sigma]$, this means some equations for hyperplanes take the form $x_v=-x_u$. Shifted by $\frac{1}{2}\mathbf{1}$, the corresponding hyperplane in $\hat{\mathscr{H}}[\Sigma]$ is $x_v+x_u=1$. We find vertices of $([0,1]^n,\hat{\mathscr{H}}[\Sigma])$, then, by combining equations of the forms $x_v=x_u$, $x_v=0$, $x_v=1$, or $x_v+x_u=1$. Combining such equations can only determine a point with coordinates 0, 1, or $\frac{1}{2}$. Therefore, the denominator of χ_Σ is 2 when Σ is unbalanced.

Summing up:

Theorem 82. *The inside-out polytope for coloring Σ has denominator* 1 *if Σ is balanced and* 2 *if it is unbalanced.* □

This theorem implies that the Ehrhart quasipolynomial—that is, the chromatic quasipolynomial—is a polynomial when Σ is balanced. It does not imply that the quasipolynomial has period 2 when Σ is unbalanced, because the period is only known to be a divisor of the denominator. However, we have formulas for $\chi_\Sigma(\lambda)$ from which we can deduce that the chromatic quasipolynomial has period 2—it is two different polynomials—for an unbalanced signed graph.

Problem 4. Prove that, for unbalanced Σ, the period of the chromatic quasipolynomial is exactly 2. Use a formula or formulas we proved previously.

Notes for 31 Mar. 2017 – Chris Eppolito.

9. Orientation

9.1. **Orientations of Signed Graphs.**

Let $\Sigma = (V, E, \sigma)$ be a signed graph. Recall that an orientation of Σ is a function τ taking the set of ends of edges of Σ to $\{+, -\}$ such that $\sigma(e{:}vw) = -\tau(v, e, \varepsilon_v)\tau(w, e, \varepsilon_w)$. A few simple things to note about orientations:

(1) We will often be a bit lax about edge ends, and simply denote (v, e, ε_v) by (v, e).
(2) Orientation is a condition at the ends of an edge. (This is clear from the definition, but bears repeating.)
(3) Links and loops have two distinct ends, but half edges have one end and loose edges have no ends.
(4) In pictures we use the following conventions at the ends of an edge:[10]

(5) Negative edges have opposing orientations at their ends, whereas positive edges have agreeing orientations at their ends:[11]

(6) The edges of a signed graph are bidirected under an orientation.

We let $\mathscr{O}(\Sigma)$ denote the set of orientations of Σ.

Definition 26. A walk $W = v_0e_1v_1 \cdots v_{l-1}e_lv_l$ in Σ is *coherent* with orientation τ when $\tau(v_i, e_{i-1})\tau(v_i, e_i) = -$ for all i. A closed walk is closed coherent with τ if this condition holds with subscripts interpreted mod l. A *cycle* is a coherently oriented closed walk.

Definition 27. An orientation τ is *acyclic* when it has no cycles, and τ is *totally cyclic* when every edge of Σ belongs to a cycle under τ.

We let $\mathscr{A}\mathscr{O}(\Sigma)$ denote the set of acyclic orientations of Σ.

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[10] In the pictures below, the signs indicate the value of τ at the indicated end of the edge and the arrows indicate the convention we use to denote this orientation at that end.

[11] In the pictures below, the signs indicate the sign on the edge and arrows indicate the orientation at the ends of the edge.

Definition 28. A pair (τ, κ) with orientation τ and coloration κ is called *proper* when for every edge of the form $e{:}vw$ we have $\tau(v,e)\kappa(v) + \tau(w,e)\kappa(w) > 0$. The pair is *compatible* when the inequality above is weak.

The following proposition is easy to prove (the proof amounts to a series of observations).

Proposition 83. *Let τ be an orientation of Σ and κ a coloration.*

(1) If κ is an improper coloration, then (τ, κ) is improper.
(2) If (tau, κ) is proper, then for an edge $e{:}vw$, $\kappa(w) > \kappa(v)$ implies $\tau(w,e) = +$.
(3) If κ is a proper coloration then there is a unique acyclic orientation τ such that (κ, τ) is proper.

There is a famous theorem of Richard Stanley's for unsigned graphs:

Theorem 84. *The number of proper pairs (κ, τ) with κ a λ-coloration of Γ is precisely $\chi_\Gamma(\lambda)$. The number of compatible pairs (κ, τ) with κ a λ-coloration is precisely $(-1)^n\chi_\Gamma(-\lambda)$.*

Zaslavsky proved that the same holds for signed graphs! (The proof is omitted due to lack of time; up to some complications the proof is the same as that for unsigned graphs.)

9.2. **Digression on Signed Digraphs.**

I will not discuss the theory of signed digraphs; the point here is only that these objects are indeed different from oriented signed graphs. One can see this in the very definition of balance.

Definition 29. A *signed digraph* is a pair (D, σ), where D is a directed graph and $\sigma : E(D) \to \{+, -\}$ is a sign function. A *directed cycle* is a loose edge of D or a circle agreeing with the directions on $E(D)$. The signed digraph (D, σ) is called *balanced* when the underlying signed graph is balanced, and *cycle balanced* when every cycle is positive.

Thus, balance implies cycle balance, but not the reverse. I do want to mention, though, the nice theorem connecting the two kinds of balance.

Theorem 85 (Harary–Norman–Cartwright [7])**.** *A strongly connected digraph with edge signs is cycle balanced if and only if it is balanced.*

The theorem fails as soon as the underlying digraph has a component that is not strongly connected.

Notes for 3 Apr. 2017 – (Josh Carey).

Notes for 5 Apr. 2017 – Amelia Mattern.

Given an orientation τ, define a mapping $\bar{R} : \tau \to \bar{R}(\tau) \subseteq \mathbb{R}^n$ by

$$\tau \mapsto \begin{cases} \{x \in \mathbb{R}^n \mid \tau(u,e)x_u + \tau(v,e)x_v > 0 \text{ for } e{:}uv, & \\ \qquad\qquad \tau(u,e) > 0 \text{ for } e{:}u\} & \text{if there are no loose edges,} \\ \varnothing & \text{if there any loose edges.} \end{cases}$$

It is clear that $\bar{R}(\tau)$, if not empty, is a region of $\mathscr{H}[\Sigma]$.

Conversely, define a mapping $\bar{\tau} : R \mapsto \tau_R$ of regions to orientations by choosing a point $x \in R$ and letting τ_R satisfy

$$\tau_R(u,e)x_u + \tau_R(v,e)x_v > 0 \text{ if } e{:}uv \in E$$

and

$$\tau_R(u,e)x_u > 0 \text{ if } e{:}u \in E.$$

Note that the choice of $x \in R$ is immaterial.

Theorem 86. *The acyclic orientations of* Σ *correspond (bijectively) to the regions of* $\mathscr{H}[\Sigma]$ *by the mapping* $\bar{R}$. *The inverse function is* $\bar{\tau}$.

Remark 2. There are no regions if Σ has a loose edge or positive loop. That is fortunate, since $\tau(u,e)x_u + \tau(v,e)x_v = [\tau(u,e) + \tau(v,e)]x_v = 0x_v = 0$ for a positive loop; therefore the inverse function would be ill-defined when Σ has a positive loop if there were any regions.

Remark 3. For $e{:}uv$, the hyperplane is $h_e : x_u - \sigma(e)x_v = 0$. Equivalently, $h_e : \sigma(e)x_u - x_v = 0$. For the former, one half-space of h_e is $x_u - \sigma(e)x_v > 0$ and the other is $x_u - \sigma(e)x_v < 0$. For the latter, one half-space of h_e is $\sigma(e)x_u - x_v > 0$ while the other is $\sigma(e)x_u - x_v < 0$. We will use only the former equivalent version of h_e and its half-spaces.

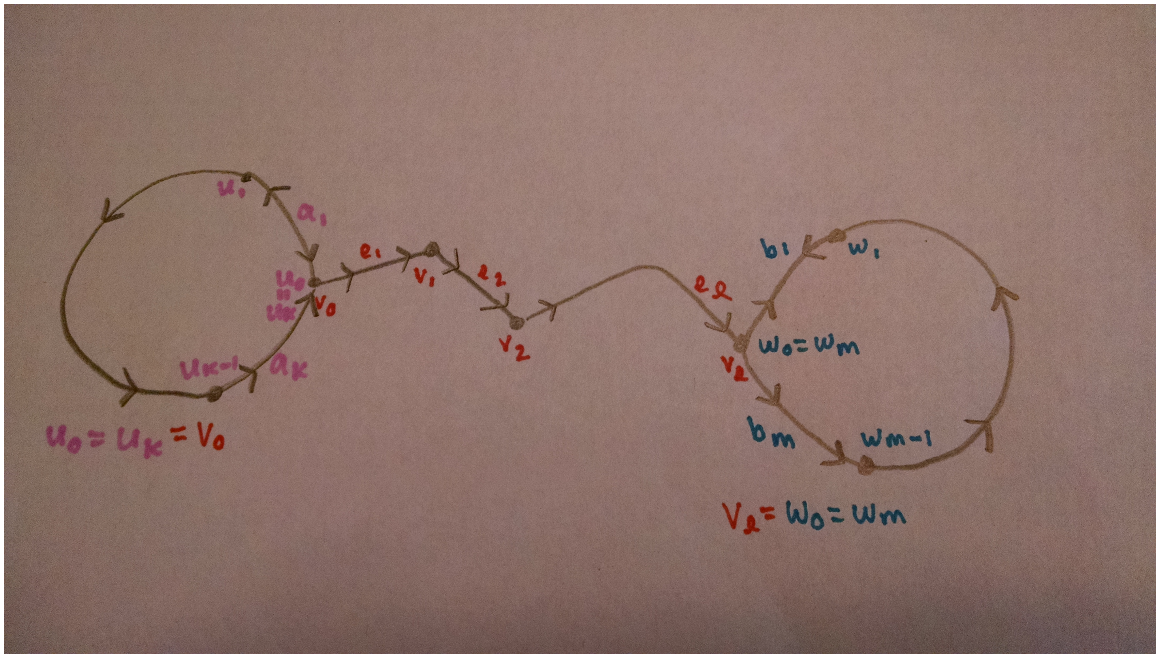
Remark 4. Suppose an edge $e{:}uv$ is oriented so that $\tau(u,e)x_u - \tau(u,e)\sigma(e)x_v = 0$, or in other words $\tau(u,e)[x_u - \sigma(e)x_v] = 0$. A point $x \notin h_e$ chooses the value $\tau(u,e)$ that makes $\tau(u,e)[x_u - \sigma(e)x_v] > 0$, in other words the value $\tau(u,e) = \operatorname{sgn}(x_u - \sigma(e)x_v)$. If an edge is oriented so that $\tau(u,e)x_u + \tau(v,e)x_v = 0$ then a similar choice is made. These are proven to be well-defined by a calculation.

Proof. First note that $\tau_{\bar{R}(\tau)} = \tau$ (if $\bar{R}(\tau) \neq \varnothing$) and $\bar{R}(\tau_R) = R$ since both function definitions have the same conditions. Also note that $\bar{R}(-\tau) = -\bar{R}(\tau)$, which shows that reversing the orientation is central reflection of the regions (hence the set of regions is centrally symmetric).

Let $\mathrm{B} = (\Sigma, \tau)$ be a bidirected cycle, so $\Sigma(\mathrm{B})$ is a positive circle or a contrabalanced handcuff. We want to show that $\bar{R}(\mathrm{B}) = \varnothing$. Note that switching $\Sigma(\mathrm{B})$ has the effect of reversing coordinate axes. In other words, $\Sigma(\mathrm{B})$ switched by vertex u changes x_u to $-x_u$ in all the formulas, so it amounts to reversing the u-axis. Therefore, we may switch as desired.

Case 1: Assume B is a positive circle $v_0e_1v_1e_2\dots v_{n-1}e_nv_n$ where $v_0 = v_n$. Switch so $\Sigma(\mathrm{B})$ is all positive. Then $\Sigma(\mathrm{B})$ is oriented $v_0 \to v_1 \to v_2 \to \dots$ (or the opposite). Thus $\tau(v_{i-1},e_i) = -$ and $\tau(v_i,e_i) = +$ for all i. So $\bar{R}(\tau)$ will have $\tau(v_{i-1},e_i)x_{i-1} + \tau(v_i,e_i)x_i > 0$. Thus $x_i > x_{i-1}$ for all i. So $x_n > \dots > x_i > x_{i-1} > \dots > x_1 > x_0 = x_n$. This is impossible. Therefore, $\bar{R}(\tau) = \varnothing$.

Case 2: Assume B is a contrabalanced handcuff and assume $\Sigma(\mathrm{B})$ is as shown in the figure below, with a_1 and b_1 the only negative edges after switching.



The path from v_0 to v_l gives us $x_{v_l} > x_{v_0}$ if $l > 0$ and $x_{v_l} = x_{v_0}$ if $l = 0$. The path from u_1 to u_k gives us $x_{u_k} > x_{u_{k-1}} > \ldots > x_{u_1}$. So $x_{u_k} > x_{u_1}$ if $k > 1$ and $x_{u_k} = x_{u_1}$ if $k = 1$. The path from w_1 to w_m gives us $x_{w_0} = x_{w_m} \leq x_{w_1}$. So $x_{u_0} \geq x_{u_1}, x_{w_0} \leq x_{w_1}$, and $x_{w_0} \geq x_{u_0}$. The negative edges give us $x_{u_0} + x_{u_1} > 0$ and $x_{w_0} + x_{w_1} < 0$. Using all of this together we get

$$x_{w_0} \geq x_{u_0} \geq x_{u_1} \text{ and } x_{u_0} + x_{u_1} > 0 \implies x_{u_0} > 0 \implies x_{w_0} > 0$$

$$x_{w_0} \leq x_{w_1} \text{ and } x_{w_0} + x_{w_1} < 0 \implies x_{w_0} < 0.$$

A contradiction! Thus $\bar{R}(\tau) = \varnothing$. □

Notes for 19 Apr. 2017 – Micah Loverro.

We assume Σ has no loose edges, since if it does things become trivial (there are no acyclic orientations and no regions). Recall that $\mathscr{R}[\Sigma]$ is the set of regions of $\mathscr{H}[\Sigma]$, $\mathscr{O}(\Sigma)$ is the set of orientations of Σ, and $\mathscr{A}\mathscr{O}(\Sigma)$ is the set of acyclic orientations of Σ.

We have the function $\bar{R} : \mathscr{O}(\Sigma) \to \mathscr{R}[\Sigma] \cup \{\varnothing\}$ given by

$$\tau \mapsto \{x \in \mathbb{R}^n \mid \tau(v_i, e)x_i + \tau(v_j, e)x_j > 0 \text{ for } e{:}v_iv_j, \tau(v_i, e)x_i > 0 \text{ for } e{:}v_i\}.$$

In the other direction we have $\mathscr{R}[\Sigma] \to \mathscr{A}\mathscr{O}(\Sigma) \subseteq \mathscr{O}(\Sigma)$, given by $R \mapsto \tau_R$ where τ_R is defined by requiring $\tau_R(v_i, e)x_i + \tau_R(v_j, e)x_j > 0$ for all $e{:}v_iv_j$ and $\tau_R(v_i, e)x_i > 0$ for all $e{:}v_i$, where x is any point in R (it doesn't matter which).

The mapping $\bar{R}$ is one-to-one since the orientation can't change without crossing a hyperplane.

Proposition 87. *The functions above satisfy*

(1) $\bar{R}(\tau_R) = R$ *and*
(2) $\tau_{\bar{R}(\tau_0)} = \tau_0$ *if* $\bar{R}(\tau_0) \neq \varnothing$.

Proof. These properties follow easily from the definitions. □

By Part (1), $\bar{R}$ is onto $\mathscr{R}[\Sigma]$. It is easy to see that the mapping $\bar{\tau} : R \mapsto \tau_R$ is injective.

Theorem 88. *The function $\bar{\tau} : \mathscr{R}[\Sigma] \to \mathscr{A}\mathscr{O}(\Sigma)$ is a bijection.*

Proof Attempt (incomplete). We need to show that every acyclic orientation is τ_R for some region R, i.e., $\bar{R}(\tau) \neq \varnothing$ if τ is acyclic.

First, assume Σ is all positive, or that we are just in the case of graphs. Each $e^+{:}v_iv_j$ corresponds to the hyperplane $h_e{:}x_i = x_j$. The orientation is either $\overrightarrow{v_iv_j}$ or $\overrightarrow{v_jv_i}$. The possibilities for a link are $\tau(v_i, e) = -1$ and $\tau(v_j, e) = +1$ so $x_j > x_i$, or $\tau(v_i, e) = +1$ and $\tau(v_j, e) = -1$ so $x_i > x_j$. For a half edge, $e{:}v_i$ is oriented toward or out of v_i, thus $x_i > 0$ or $x_i < 0$.

Let τ be an acyclic orientation of $+\Gamma$. For each $e{:}v_iv_j$ we have $v_i > v_j$ or $v_i < v_j$, and for each $e{:}v_i$ we have $v_i > 0$ or $v_i < 0$.

Lemma 89. *This relation extends to a partial ordering of $V \cup \{0\}$.*

Proof. Extend $>$ by transitivity. Suppose to the contrary we have a reflexivity $v_1 < v_1$. That means there exist $v_1 < v_2 < \cdots < v_p < v_1$ in V. Choose p minimal, so there are no repeats among the v_i. Therefore we have a cycle in (Γ, τ), contrary to the hypothesis. If $0 < 0$, there exist $0 > v_1 > \cdots > v_p > 0$, so we still get a cycle. Therefore, no reflexivity exists, and that implies no symmetric pair $v_i > v_j > v_i$ exists. Thus, we have a partial ordering. $\square$

Proposition 90. *Let Γ be a graph.*

1. *An acyclic orientation of Γ determines a partial ordering of V.*
2. *A partial ordering of V determines an acyclic orientation of Γ by orienting $e{:}v_iv_j$ from the lesser vertex to the greater vertex.*

Proof. Part 1 is shown in the lemma. For Part 2, there are no cycles because there are no downward edges. $\square$

This correspondence is not a bijection, as multiple partial orderings may give the same orientation.

Now we complete the proof of [**Theorem 88? Is it the right theorem?**] Given $\tau \in \mathscr{A}\mathscr{O}(\Gamma)$, define $x \in \bar{R}(\tau)$ as follows. Choose a linear extension of the partial ordering defined by τ. By choice of names, we may assume $v_1 < v_2 < \cdots < v_n$. Let $x_i = i$. Then $x \in \bar{R}(\tau)$. Since $\bar{R}(\tau)$ is nonempty, it is a region. $\square$

Notes for 21 Apr. 2017 – Ted Ofner.

In this lecture we attempted to prove that every acyclic orientation α of a signed graph Σ produces a corresponding region $R(\alpha)$; i.e., $R(\alpha)$ is non-empty.

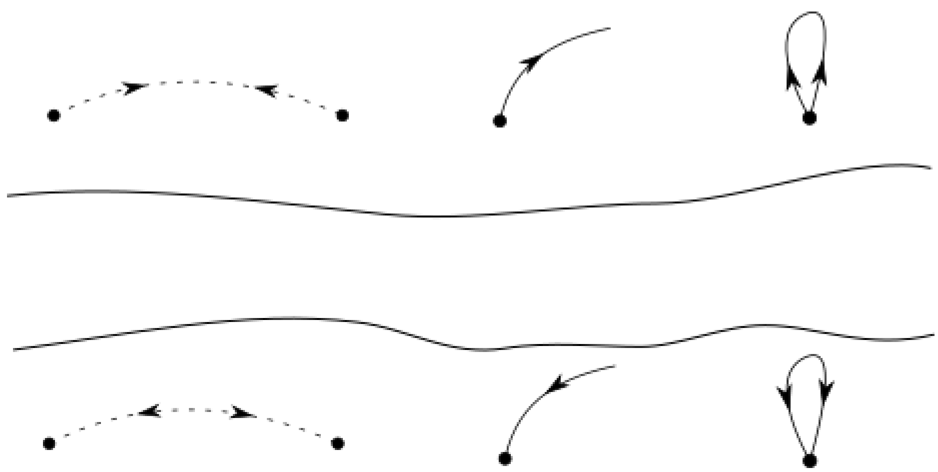
Recall that for an unsigned graph (in signed graph terms, an all positive graph) Γ we proved that an acyclic orientation α of Γ induces a well formed (strict) partial order $<_\alpha$ on $V(\Gamma) \cup \{0\}$ such that all oriented edges point in an increasing direction. This allowed us to show the corresponding region $R(\alpha)$ was empty by constructing a point in $\mathbb{R}^V$ with coordinates values satisfying the inequalities of their corresponding vertices.

Turning to general signed graphs, we begin by assuming Σ is switched in such a way that Σ^+ is a connected spanning subgraph. Our recalled fact gives a partial ordering $<_\alpha$

on $V(\Sigma^+) \cup \{0\} = V(\Sigma) \cup \{0\}$. We hope to add in our negative edges without creating any contradictions.

Consider the inequalities implied by a negative edge. The two posibilities for a negative edge are shown in the diagram below:

In the context of the regions, Type 1 would induce the inequality $x_u + x_v < 0$; Type 2 would induce the inequality $x_u + x_v > 0$. Half edges induce $x_u < 0$ or $x_u > 0$ depending on the orientation, and negative loops act the same as half edges. After some thought and discussion, we arrived at the following general scenario which captures the different ways an orientation could produce a contradiction in its induced inequalities:



This diagram describes the 9 disctinct cases which result in contradictory induced inequalities. In each case, an item from the top row is connected to an item in he bottom row by a path or paths connecting the marked vertices. All connecting paths are oriented coherently pointing from the bottom vertices to the top vertices. The paths are not assumed to be internally disjoint, since no matter which case is constructed, an oriented cycle can be found as a subgraph. As such, an acyclic orientation of Σ can be seen to induce no contradictory inequalities on its region $R(\alpha)$.

Having established that our induced inequalities were non-contradictory, however, was not enough for us to actually construct a point $x \in R(\alpha)$. We discussed for a bit, but failed to arrive at a solution.

Notes for 24 Apr. 2017 – Chris Eppolito.

Let $\Sigma = (V, E, \sigma)$ be a signed graph without loose edges and positive loops. Recall that a pair (α, κ) in which α is an orientation of Σ and κ is a coloration of Σ is **compatible** when the inequalities below are all satisfied:

$$\begin{aligned} \alpha(v)\kappa(v) + \alpha(w)\kappa(w) &\geq 0 && \text{for } e{:}vw \in E, \\ \alpha(v)\kappa(v) &\geq 0 && \text{for } e{:}v \in E. \end{aligned}$$

A compatible pair is **proper** when the inequalities above are satisfied strictly. We seek to prove the following theorem over the next several meetings:

Theorem 91 (Signed Stanley Theorem)**.** *The number of proper pairs (α, κ) in which κ is any λ-coloration equals $\chi_\Sigma(\lambda)$. Furthermore, the number of compatible pairs equals $(-1)^{\#V}\chi_\Sigma(-\lambda)$.*

In what remains we set the stage with the ingredients for a proof.

Primarily, notice that bidirection of (Σ, α) and acyclicity of α imply the existence of a unique canonical incidence matrix $\mathrm{H} = \mathrm{H}(\Sigma) = (\eta_{v,e})_{v \in V, e \in E}$ for Σ. For an edge $e \in E$, let η_e denote the column of H corresponding to the edge e. We collect several simple facts concerning this notation.

1. The entry η_{ve} of $\mathrm{H}(\Sigma)$ satisfies

$$\eta_{ve} = \sum_{\text{incidences } (v,e)} \alpha(v,e) = \begin{cases} 0 & \text{if } e \text{ is not incident to } v, \text{ or } e{:}vv \text{ is a positive loop,} \\ \pm 1 & \text{if } e{:}vw \text{ for } w \neq v, \\ \pm 2 & \text{if } e{:}vv \text{ is a negative loop.} \end{cases}$$

2. The hyperplane h_e equals $\eta_e^\perp = \{x \in \mathbb{R}^V \mid \eta_e \cdot x = 0\}$.
3. The closed half-space $h_e^{\geq}$ equals $\{x \in \mathbb{R}^V \mid \eta_e \cdot x \geq 0\}$.
4. The open half-space $h_e^{>}$ equals $\{x \in \mathbb{R}^V \mid \eta_e \cdot x > 0\}$.

Recall that for use in the Ehrhart theory of graph coloring we defined the shifted hyperplane arrangement of Σ to be

$$\mathscr{H}^\sharp[\Sigma] = \{h_e + \tfrac{1}{2}\mathbf{1} \mid e \in E\}.$$

We collect some remarks from previous lectures in the following list. Let $\lambda \in \mathbb{Z}_{\geq 0}$. Then

1. The proper λ-colorations of Σ correspond to the $\frac{1}{\lambda+1}$-fractional points of $(0,1)^V \setminus \bigcup \mathscr{H}^\sharp[\Sigma]$.
2. All λ-colorations correspond to the $\frac{1}{\lambda-1}$-fractional points in $[0,1]^V$.
3. (Theorem 81) The open Ehrhart quasipolynomial and chromatic quasipolynomial are related by
$$E^\circ_{[0,1]^V, \mathscr{H}^\sharp[\Sigma]}(\lambda+1) = \chi_\Sigma(\lambda).$$

Finally, recall the following theorems from earlier in the course.

Theorem 92. *The acyclic orientations of Σ are in natural bijection with the regions of the hyperplane arrangement $\mathscr{H}[\Sigma]$.*

Theorem 93. *For $A \subseteq E$,*

$$\mu_{\mathrm{Lat}(\Sigma)}(\varnothing, A) = \sum_{\substack{S \subseteq A \\ \mathrm{clos}(S) = A}} (-1)^{\#S}.$$

Theorem 94. *The chromatic polynomial of* Σ *satisfies*

$$\chi_\Sigma(\lambda) = \sum_{A \in \mathrm{Lat}(\Sigma)} \mu(\varnothing, A)\lambda^{b(A)}$$

for all odd positive integers λ.

Theorem 95. *The Möbius function of* $\mathrm{Lat}(\Sigma)$ *alternates in sign.*[12] *Specifically,*

$$(-1)^{\mathrm{rk}(A)}\mu(\varnothing, A) \geq 0$$

(and it is positive if $\varnothing$ *is closed, i.e.,* Σ *has no positive loops or loose edges.*

Theorem 96 (Ehrhart Reciprocity)**.** *The Ehrhart quasipolynomial of a polytope* Q *satisfies*

$$E_Q(k) = (-1)^{\dim(Q)} E_Q^\circ(-k)$$

for every $k \in \mathbb{Z}_{\geq 0}$.

Theorem 97. *The Ehrhart quasipolynomial of an inside-out polytope* $(P, \mathscr{H})$ *satisfies*[13]

$$E_{P,\mathscr{H}}(k) = \sum_{s \in \mathscr{L}(\mathscr{H})} \mu_{\mathscr{L}(\mathscr{H})}(\varnothing, s) E_{P \cap s}(k).$$

To prove the Signed Stanley Theorem (Theorem 91), we need the following proposition. For a λ-coloration κ considered as a point in $[0,1]^V$ as above, let $t(\kappa)$ denote the intersection of all hyperplanes $h \in \mathscr{H}^\sharp[\Sigma]$ such that $\kappa \in h$.

Proposition 98. *Each* λ*-coloration* κ *belongs to precisely*

$$c(\kappa) = \sum_{\substack{s \in \mathscr{L}(\mathscr{H}[\Sigma]) \\ t(\kappa) \subseteq s}} |\mu_{\mathscr{L}(\mathscr{H}[\Sigma])}(\varnothing, s)|$$

closed regions of $\mathscr{H}^\sharp[\Sigma]$.

We will prove this over the next few lectures.

Notes for 26 Apr. 2017 – Josh Carey.

The number of $\frac{1}{\lambda+2}$-fractional lattice points in $[0,1]^n \cap u$ for $u \in \mathscr{L}(\mathscr{H}^\sharp[\Sigma])$ is $E_{[0,1]^n \cap u}(t) = (\lambda+1)^{\dim u}$, because if we project u onto a face of $[0,1]^n$ of dimension $\dim u$, the projection is a bijection of lattice points. To get the number we apply the main theorem of inside-out

[12] This theorem is due to Rota. It holds more generally for semimodular lattices. See Stanley, *Enumerative Combinatorics*, Volume I.

[13] This is the central theorem of inside-out Ehrhart theory. See Beck and Zaslavsky, "Inside-Out Polytopes".

theory (the first equality following):

$$\begin{aligned}
E^0_{[0,1]^n,\mathscr{H}^\sharp}(-\lambda+1) &= \sum_{u\in\mathscr{L}(\mathscr{H}^\sharp[\Sigma])} \mu_{\mathscr{L}}(\varnothing,u)E^0_{[0,1]^n\cap u}(-\lambda+1)\\
&= \sum_{A\in\operatorname{Lat}\Sigma} \mu(\varnothing,A)\lambda^{b(A)}(-1)^{b(A)}\\
&= \sum_{A\in\operatorname{Lat}\Sigma} (-1)^{n-b(A)}|\mu(\varnothing,A)|\lambda^{b(A)}(-1)^{b(A)}\\
&= (-1)^n \sum_{A\in\operatorname{Lat}\Sigma} |\mu(\varnothing,A)|\lambda^{b(A)}.
\end{aligned}$$

Now we use a theorem of Rota's to show that we have here an evaluation of the chromatic polynomial.

Theorem 99 (Rota)**.** $|\mu(\varnothing,A)| = (-1)^{\operatorname{rk}A}\mu(\varnothing,A) = (-1)^n[(-1)^{b(A)}\mu(\varnothing,a)]$.

So we can replace signs in the chromatic polynomial by absolute values:

$$\begin{aligned}
\chi_\Sigma(-\lambda) &= \sum_{A\in\operatorname{Lat}\Sigma} \mu(\varnothing,A)(-1)^{b(A)}\lambda^{b(A)}\\
&= \sum_{A\in\operatorname{Lat}\Sigma} (-1)^n|\mu(\varnothing,A)|\lambda^{b(A)}\\
&= (-1)^n \sum_{A\in\operatorname{Lat}\Sigma} |\mu(\varnothing,A)|\lambda^{b(A)}.
\end{aligned}$$

Shifting the sign factor to the other side, we get the formula we were after:

$$\begin{aligned}
(-1)^n\chi_\Sigma(-\lambda) &= \sum_{A\in\operatorname{Lat}\Sigma} |\mu(\varnothing,A)|\lambda^{b(A)}\\
&= E^0_{[0,1]^n,\mathscr{H}^\sharp}(-\lambda+1).
\end{aligned}$$

Disclaimer: There are some errors in these notes which were corrected in the subsequent lecture.

Notes for 28 Apr. 2017 – Amelia Mattern.

Summarizing essential facts of Ehrhart theory:

1. $E^\circ_Q(t) = (-1)^{\dim Q}E_Q(-t)$ for a rational polytope Q and an integer t.
2. $E^\circ_{Q\cap\mathscr{H}}(t) = (-1)^{\dim Q}E_{Q,\mathscr{H}}(-t)$ for a rational hyperplane arrangement $\mathscr{H}$.
3. $E^\circ_{P,\mathscr{H}}(t) = \sum_{s\in\mathscr{L}(\mathscr{H})}\mu(\hat{0},s)E^\circ_{P\cap s}(t)$ for $P=[0,1]^n$ in $\mathbb{R}^n$.

A main consequence of Theorem 97 is its closed version.

Corollary 100. *For a rational hyperplane arrangement $\mathscr{H}$ and rational polytope P in $\mathbb{R}^n$,*

$$E_{P,\mathscr{H}}(t) = \sum_{s\in\mathscr{L}(\mathscr{H})}(-1)^{\operatorname{codim}s}\mu(\hat{0},s)E_{P\cap s}(t) = \sum_{s\in\mathscr{L}(\mathscr{H})}|\mu(\hat{0},s)|E_{P\cap s}(t).$$

Our P will be $[0,1]^n$ but this is general, provided that P is full-dimensional and every hyperplane of $\mathscr{H}$ intersects the polytope's interior, P°.

Proof. We assume the dimension of P is n. We have the following equalities:

$$\begin{aligned}
E_{P,\mathscr{H}}(t) &= (-1)^n E^\circ_{P,\mathscr{H}}(-t) \\
&= (-1)^n \sum_{s\in\mathscr{L}(\mathscr{H})} \mu(\hat{0},s) E^\circ_{P\cap s}(-t) \\
&= (-1)^n \sum_{s\in\mathscr{L}(\mathscr{H})} \mu(\hat{0},s)(-1)^{\dim P\cap s} E_{P\cap s}(t) \\
&= (-1)^n \sum_{s\in\mathscr{L}(\mathscr{H})} (-1)^{\dim s}\mu(\hat{0},s) E_{P\cap s}(t) \\
&= \sum_{s\in\mathscr{L}(\mathscr{H})} (-1)^{\operatorname{codim} s}\mu(\hat{0},s) E_{P\cap s}(t) \\
&= \sum_{s\in\mathscr{L}(\mathscr{H})} |\mu(\hat{0},s)| E_{P\cap s}(t),
\end{aligned}$$

since $\operatorname{codim} s = \operatorname{rk} s$ and $\operatorname{sgn}\mu(\hat{0},s) = (-1)^{\operatorname{rk} s}$. □

Now we let $P = [0,1]^n$ in $\mathbb{R}^n$ with $\mathscr{H}^\sharp[\Sigma]$ as previously defined. Let $s \in \mathscr{L}(\mathscr{H}^\sharp[\Sigma])$. Let $Q = P \cap s$ and let $\mathscr{H}$ be the induced arrangement in s, which is

$$\mathscr{H}^{\sharp s} := \{h \cap s : h \in \mathscr{H}^\sharp[\Sigma] \text{ and } s \not\subseteq h\}.$$

The principal fact of Ehrhart theory applied to signed graphs, previously proved, is:

4. $E^\circ_{P,\mathscr{H}^\sharp[\Sigma]}(\lambda+1) = \chi_\Sigma(\lambda)$ for odd λ.

We now present our proof of the Signed Stanley Theorem, Theorem 91.

Proof. We have an integer $\lambda > 0$. We know that

$$\chi_\Sigma(-\lambda) = E^\circ_{P,\mathscr{H}^\sharp[\Sigma]}(-\lambda+1) = (-1)^n E_{P,\mathscr{H}^\sharp[\Sigma]}(\lambda-1)$$

by reciprocity. We do this in order to count the fractional lattice points κ. The latter sum

$$= (-1)^n \sum_{s\in\mathscr{L}(\mathscr{H}^\sharp[\Sigma])} |\mu(\hat{0},s)| E_{P\cap s}(\lambda-1),$$

which gives us the number of $\frac{1}{\lambda-1}$-fractional points in $P\cap s$. We rewrite this as a sum over individual lattice points κ so we can reverse the order of summation:

$$(-1)^n \sum_{s\in\mathscr{L}(\mathscr{H}^\sharp[\Sigma])} \sum_{\kappa\in s} |\mu(\hat{0},s)| = (-1)^n \sum_{\kappa}\sum_{s\ni\kappa} |\mu(\hat{0},s)|.$$

Note that we are using the color set $\{0, \frac{1}{\lambda-1}, \frac{2}{\lambda-1}, \dots, \frac{\lambda-1}{\lambda-1}\}$, which has a total of λ colors—just the right number. Now let $t(\kappa)$ be the smallest flat of $\mathscr{H}^\sharp[\Sigma]$ that contains κ, i.e., $t(\kappa) := \bigcap\{s \in \mathscr{L} : s \ni \kappa\}$. So

$$\chi(\lambda) = (-1)^n \sum_{\kappa}\sum_{s\ni\kappa} |\mu(\hat{0},s)| = (-1)^n \sum_{\kappa} \sum_{\substack{s\supseteq t(\kappa)\\ (s\leq t(\kappa))}} |\mu(\hat{0},s)|,$$

where $s \supseteq t(\kappa) = \{s \in \mathscr{L}(\mathscr{H}^\sharp[\Sigma]) : s \supseteq t(\kappa)\}$. Define $\mathscr{H}(u) = \{h \in \mathscr{H} : h \supseteq u\}$. Then $s \in \mathscr{L}(\mathscr{H})$ contains u if and only if $s \in \mathscr{L}(\mathscr{H}(u))$. Thus,

$$\sum_{s \supseteq t(\kappa)} |\mu(\hat{0}, s)| = \sum_{s \in \mathscr{H}^\sharp[\Sigma](t(\kappa))} (-1)^{\operatorname{codim} s} \mu(\hat{0}, s).$$

Recall that the characteristic polynomial of $\mathscr{H}$ has the formula

$$p_{\mathscr{H}}(\theta) = \sum_{s \in \mathscr{L}(\mathscr{H})} \mu(\hat{0}, s)\theta^{\dim s} = (-1)^n \sum_{s \in \mathscr{L}(\mathscr{H})} \mu(\hat{0}, s)\theta^{\operatorname{codim} s}.$$

Therefore,

$$\sum_{s \in \mathscr{H}^\sharp[\Sigma](t(\kappa))} (-1)^{\operatorname{codim} s} \mu(\hat{0}, s) = (-1)^n p_{\mathscr{H}^\sharp[\Sigma](t(\kappa))}(-1),$$

where -1 is the number of regions of $\mathscr{H}^\sharp[\Sigma](t(\kappa))$.

All of these regions D contain $t(\kappa)$ so they all contain κ. But they are cut up by $\mathscr{H}^\sharp[\Sigma] \setminus \bigcup \mathscr{H}^\sharp[\Sigma](t(\kappa))$ into subregions R that are regions of $\mathscr{H}^\sharp[\Sigma]$. Since κ is not in any of those left-out hyperplanes it is in only one of those subregions R for each D. Therefore, κ is in exactly $(-1)^n p_{\mathscr{H}^\sharp[\Sigma](t(\kappa))}(-1)$ closed regions of $\mathscr{H}^\sharp[\Sigma]$. Equivalently, κ is compatible with that number of acyclic orientations. It follows that

$$\begin{aligned}\chi_\Sigma(-\lambda) &= (-1)^n \sum_\kappa (\text{the number of acyclic orientations compatible with } \kappa) \\ &= (-1)^n (\text{the number of compatible pairs altogether}). \qquad \square\end{aligned}$$

Notes for 1 May 2017 – Micah Loverro.

11. No Two Disjoint Negative Circles—with Projective Planarity

We now turn to a combinatorial problem about signed graphs that has a topological answer.

Since negative circles are fundamental to signed graphs, a natural question to ask is: Which signed graphs have no two disjoint negative circles? The answer is due to Lovász and Slilaty.

Theorem 101. *The signed graphs with no two disjoint negative circles are the following:*

(1) *Projective-planar signed graphs,*
(2) *signed graphs Σ with a balancing vertex v,*
(3) *some others, and*
(4) *certain simple balanced extensions of the above.*

(The theorem is stated precisely and completely in Theorem 113.)

Graphs of type (2) clearly have the property. Since any circle in $\Sigma \setminus v$ is positive, any two negative circles in Σ must have at least v in common.

(Recall that the subgraphs that give minimal dependencies (i.e., circuits in the matroid) are (i) the contrabalanced handcuffs (two disjoint negative circles connected by a path, or two negative circles touching only at a vertex, and (ii) a positive circles. Thus this leads naturally to a slightly different question: Which signed graphs contain no subgraphs of type (i)? Then the only circuits would be the positive circles. This question has a very different answer; see [19].)

11.1. **Surface embedding of signed graphs.**

We now consider embeddings of signed graphs into closed surfaces. Throughout, we assume that Σ has no loose edges or half-edges. [For these embeddings see [17].]

Let T_g be the orientable closed surface of genus g. It is the sphere with g handles; that is, the connected sum of a sphere with g tori. Let U_h be the connected sum of h copies of the projective plane $\mathbb{P}^2$. We also view T_g as a sphere with g handles and U_h as sphere with h cross-caps. Recall that the Euler characteristic of T_g is $2 - 2g$ and the demigenus is $2g$. The Euler characteristic of U_h is $2 - h$ and the demigenus is h. A handle can be added as a *prohandle* or an *antihandle*. They correspond respectively to the connected sum with a torus or Klein bottle. Connected sum induces a partial order on closed surfaces: $A \leq B$ if B can be constructed by attaching handles or cross-caps to A.

If we allow graph embeddings which are not cellular, then an embedding in a smaller surface automatically gives an embedding in any larger surface since we are free to attach additional handles or cross-caps in any of the regions. A natural question is then: Given a graph Γ, what is the smallest surface for which there exists an embedding?

For this we need a combinatorial description of an embedded graph, or at least a cellularly embedded graph. A *rotation* at a vertex v_i as a cyclic permutation edge ends incident to v_i. A *rotation system* for a graph is a function

$$\rho : V \to \{\text{cyclic permutations of edge ends}\}$$

such that each $\rho(v_i)$ is a rotation at v_i.

Rotations can be regarded as permuting the neighboring vertices if (and only if) the graph is simple. For example, if v_1 is adjacent to vertices v_2, v_5 and v_9, with no loops or multiple edges involving v_1, then we can regard the rotation at v_1 as $(v_2v_5v_9)$ or $(v_2v_9v_5)$. If there are loops or multiple edges, we need to distinguish further since an edge end incident to v_i is not determined by its adjacent vertex.

Theorem 102. *The orientable cellular embeddings of a graph Γ correspond bijectively to the rotation systems on Γ.*

This is a special case of a signed-graph generalization. A *rotation system* for a signed graph Σ is a rotation system for $|\Sigma|$.

Proposition 103. *Given an orientation embedding $\Sigma \hookrightarrow S$ of a balanced signed graph into a closed surface S, there is a cellular embedding $\Sigma \hookrightarrow T_g \leq S$ into an orientable surface T_g smaller than or equal to S.*

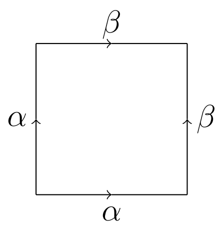
Proof Sketch. The initial orientation embedding gives us a rotation system. Consider a walk starting at some vertex v_0, and choose an edge incident to v_0 to start with. From the next vertex v_1, follow the rotation at v_1 and use the next edge end in the rotation at v_1 to extend the walk. We need to make a modification if we arrive at a vertex v_i by following a negative edge; then we use $\rho^{-1}(v_i)$ instead of $\rho(v_i)$. Given the initial choice of vertex and edge, this gives a well-defined walk which will eventually return to v_0. The final step is to glue 2-cells along each such walk. □

Notes for 3 May 2017 – Chris Eppolito.

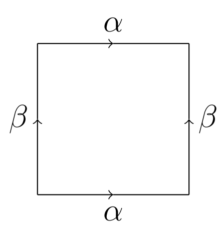
Now I assume the reader is familiar with the classic theorem that classifies all closed, connected surfaces (the only kind we need to consider).[14]

Theorem 104 (Classification of Closed Surfaces)**.** *Every compact, connected surface without boundary is a connected sum of tori and projective planes. Every compact, connected surface without boundary can be expressed as a quotient space of a closed polygonal disk by pairwise identifications of edges.*

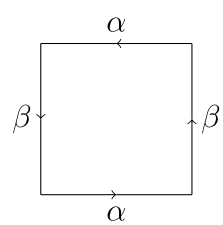
Thus we can draw planar diagrams of surfaces. Here are some typical examples:



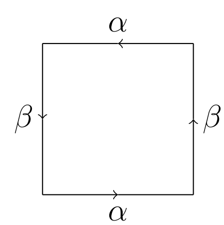
Sphere
$T_0 = \mathbb{S}^2$



Torus
T_1



Projective Plane
$\mathbb{P}^2$



Klein Bottle
U_2

There are more efficient ways to draw some of these diagrams ....
We use the following notations:

(1) The orientable surface of genus g is denoted by T_g.
(2) The unorientabe surface of demigenus d is denoted by U_d.
(3) The symbol $\#$ denotes connected sum of surfaces. That means taking a disk in each surface, cutting out the interior, and identifying the two boundary circles.
(4) Surfaces $S \leq T$ iff $T \cong S \# S'$ for some surface S'.

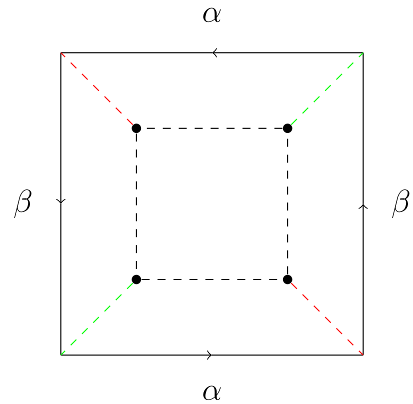
The theory of graph embeddings is well studied; *Graph Theory* by Bondy and Murty and *Graphs on Surfaces* by Mohar and Thomassen are good references for the elementary theory. We now develop a small portion of this theory for signed graphs.

Definition 30. An *orientation embedding* of a signed graph Σ (assumed to have no loose or half edges) in a surface S is an embedding of $|\Sigma|$ in S such that a circle C of Σ is positive if and only if the embedded C is orientation preserving in S. We write $\Sigma \hookrightarrow S$ to denote Σ admits an orientation embedding into S.

(In graph embedding we only consider graphs without loose and half edges. Loose edges embed as contractible circles separate from the rest of the graph and half edges require boundary.)

[14]*Algebraic Topology: An Introduction* by William S. Massey is a good reference.

Example 7. The following diagram depicts an orientation embedding $-K_4 \hookrightarrow \mathbb{P}^2$ via planar diagram:



We collect several simple facts about orientation embeddings in Proposition 106. First we should define link minors

Definition 31. A *link minor* of a signed graph Σ is a minor of Σ such that when expressed as a series of single-edge contractions followed by a deletion, each contraction is a contraction of a link.

Proposition 105. *A minor of Σ is a link minor if and only if it is obtained from a subgraph of Σ by contracting a forest.*

Proof. Exercise. □

Proposition 106. *Let Σ be a signed graph and let S be a surface.*

1. *If $\Sigma \hookrightarrow S$, then also all switchings $\Sigma^\zeta \hookrightarrow S$.*
2. *If $\Sigma \hookrightarrow S$, then $\Sigma \setminus e \hookrightarrow S$ for all $e \in E$.*
3. *If $\Sigma \hookrightarrow S$, then $\Sigma / e \hookrightarrow S$ for all $e \in E$.*
4. *The class of signed graphs that orientation-embed in S is closed under taking link minors.*
5. *If $\Sigma \hookrightarrow S$ and $S \leq T$, then $\Sigma \hookrightarrow T$.*

Proof. Exercise. These proofs mostly mimic the proofs of the corresponding statements for unsigned graphs. □

Under our partial order on surfaces, $T_g < U_h$ if and only if $2g < h$. This fact can be leveraged to prove the following simple results. The surface $S(\Sigma)$ in part 2 is called the *minimal surface* of Σ. We let $d(\Sigma)$ denote the demigenus of $S(\Sigma)$.

Proposition 107. *Let Σ be a signed graph.*

1. *If Σ is balanced and $\Sigma \hookrightarrow U_d$, then $\Sigma \hookrightarrow T_{\lfloor (d-1)/2 \rfloor}$.*
2. *There is a surface $S(\Sigma)$ such that $\Sigma \hookrightarrow T$ if and only if $T \geq S(\Sigma)$.*
3. *The minimal surface $S(\Sigma)$ is a switching-isomorphism invariant of Σ.*
4. *If Σ is balanced, then $S(\Sigma) = T_g$ for some genus g.*
5. *If Σ is unbalanced, then $S(\Sigma) = U_h$ for some demigenus h.*

Proof. Exercise. □

Corollary 108. *The demigenus $d(\Sigma)$ and the state of balance of Σ determine $S(\Sigma)$.*

Of particular interest for us will be the projective-planar signed graphs. These are the signed graphs that orientation-embed in the real projective plane; the latter may be notated U_1 or $\mathbb{P}^2$ but in our context it suffices to write $\mathbb{P}^2$.

Lemma 109. *If $\Sigma \hookrightarrow \mathbb{P}^2$, then any two negative circles of Σ intersect in at least one vertex.*

Proof. Suppose $i : \Sigma \hookrightarrow \mathbb{P}^2$ is an orientation embedding of Σ. If Σ is balanced, then the statement holds trivially. Otherwise, choose a negative circle $C = v_0 e_1 v_1 \cdots v_{l-1} e_l v_l$ of Σ. As C is negative, we know that the orientation reverses when travelling along $i(C)$. In particular, we may draw a planar diagram of $\mathbb{P}^2$ having boundary a doubled $i(C)$ with the antipodal action for identification. Now any negative circle C' in Σ must reverse the orientation of a frame when travelling along $i(C')$; in particular, $i(C')$ must cross the boundary of any planar diagram for Σ an odd number of times. Thus $i(C') \cap i(C) \neq \varnothing$ yields C and C' must have a common vertex as i is an embedding of $|\Sigma|$. $\square$

11.2. **Forbidden Link Minors for Orientation Embedding.**

Kuratowski's Theorem for planarity states that an unsigned graph Γ is planar (i.e., $S(+\Gamma) = T_0$) if and only if the minors of Γ include neither $K_{3,3}$ nor K_5. A vast generalization of this theorem is the next statement.

Theorem 110 (The Kuratowski Theorem for Surfaces: Robertson–Seymour [11], Bodendieck–? [3])**.** *For each surface S there is a finite set $\mathbf{Ex}(S)$ of graphs such that Γ embeds in S if and only if no minor of Γ belongs to $\mathbf{Ex}(S)$.*

A similar result may hold for signed graphs.

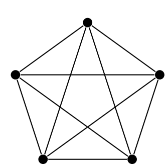
Theorem 111 (Hopeful Theorem for Orientation Embedding)**.** *For every surface S, there is a finite set $\mathbf{Ex}^{\pm}(S)$ of signed graphs such that $\Sigma \hookrightarrow S$ if and only if no link minor of Σ belongs to $\mathbf{Ex}^{\pm}(S)$.*

This is unproved in general, but in a paper entitled "The projective-planar signed graphs" [18], Zaslavsky obtained a forbidden link minor characterization for projectively orientation embeddable signed graphs.

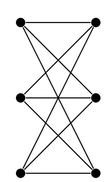
Theorem 112. *A signed graph is projective-planar if and only if it has no link minor which is switching isomorphic to any of $-\bar{K}_2^\circ$, $+K_5$, $+K_{3,3}$, Φ_4, $-W_4$, or Ψ_5.*

The signed graphs in Theorem 112 are shown below:

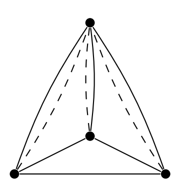
$-\bar{K}_2^\circ$



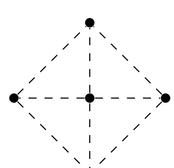
$+K_5$



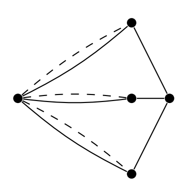
$+K_{3,3}$



Φ_4



$-W_4$



Ψ_5

Note that Φ_4 and Ψ_5 are obtained one from the other by a balanced Y-Δ transform, where a positive triangle is transformed into a 3-edge star, or vice versa. (The signs on the Y, i.e. the star edges, do not matter in Ψ_5.)

11.3. **No Non-Intersecting Negative Circles.**

Now we come to the combinatorial purpose for which I introduced projective planarity: the classification of signed graphs in which there exist no two disjoint negative circles (that is, naving nothing in common, not even one vertex).

Definition 32. For an integer $t \geq 0$, a *t-sum* of signed graphs Σ_1 and Σ_2 is obtained by fixing two embeddings $\varepsilon_i : K_t \to |\Sigma_i|$ such that $\mathrm{Im}(\varepsilon_i)$ is balanced for $i \in [2]$, switching these embedded copies of K_t all positive in both Σ_i, identifying $\mathrm{Im}(\varepsilon_1)$ with $\mathrm{Im}(\varepsilon_2)$ as prescribed by these embeddings, and finally deleting all edges of $\mathrm{Im}(\varepsilon_i)$.

The 0-sum is merely disjoint union. A 1-sum is the identification of two vertices, one in each graph. A 2-sum is the identification (after switching as necessary) of two positive edges.

Our main interest in t-summation is the following result, partially proved by Lovász (unpublished) and fully proved by Slilaty [13].

Theorem 113 (Lovász–Slilaty)**.** *A signed graph Σ has no two disjoint negative circles if and only if one of the following holds:*

(1) *Σ is balanced.*
(2) *Σ has a balancing vertex.*
(3) *Σ is projective planar.*
(4) *Σ is switching isomorphic to $-K_5$.*
(5) *Σ may be obtained by* 0-, 1-, 2-, *and* 3-*summation of balanced signed graphs with a graph of type* (3) *or* (4).

Necessity is not difficult; it will be proved in the next lecture. Sufficiency is highly nontrivial and its proof will be omitted from these notes.

Notes for 4 May 2017 – Josh Carey.

We are concerned with the signed graphs having the smallest negative-circle packing number $\nu(\Sigma)$. $\nu = 0$ means there is no negative circle; that is, Σ is balanced. The first nontrivial case is $\nu = 1$.

Theorem 114 (Lovász–Slilaty)**.** *The negative-circle packing number $\nu(\Sigma) \leq 1$ if and only if Σ arises from:*

1. *A balanced graph, or*
2. *a graph with a balancing vertex, or*
3. *a projective-planar graph, or*
4. *$-K_5$ (which is not projective planar) by:*
5. *any number of t-summations with a previously listed graph, for $t \leq 3$.*

Here are a few remarks on the proof that this construction gives only signed graphs with $\nu \leq 1$.

1. This is trivial; and since the t-sums remain balanced they are unnecessary.
2. This is easy; and the t-sums still have a balancing vertex so they are unnecessary.

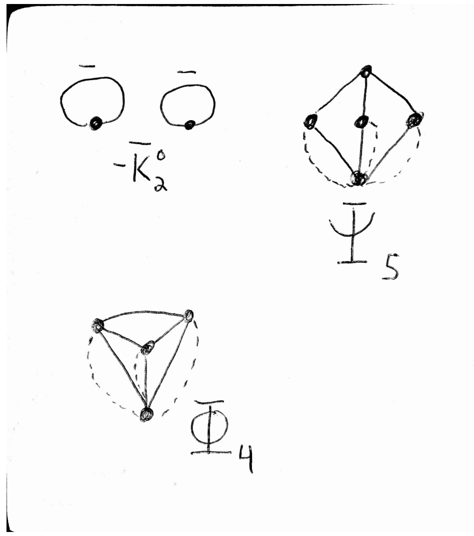


FIGURE 11.1. The forbidden link minors for orientation embeddability in the projective plane.

3. This is proved by topology: the projective plane contains no two disjoint uncontractible closed curves.
4. This is proved by inspection since K_5 contains no two disjoint circles at all; and here $t \leq 2$ because $-K_5$ does not contain a balanced K_3.
5. This one has been set as an exercise.

Theorem 115 (Zaslavsky [18]). *A signed graph Σ is projective planar if and only if it does not contain as a link minor the following graphs: $-\overline{K}_2^{\circ}$, $+K_5$, $+K_{3,3}$, $-W_4$, Ψ_5, or Φ_4.*

The less familiar graphs here are illustrated in Figure 11.1.

A similar result from long before is the following purely graphic theorem. Lovász was 16 years old when he published this (in Hungarian).

Theorem 116 (Lovász). *A graph Γ has no two disjoint circles if and only if it is one of the types you can find in English in Bollobás' book* Extremal Graph Theory*.*

Notes for 5 May 2017 – Ted Ofner.

12. Integral Flows

12.1. **The Double Covering Graph.**

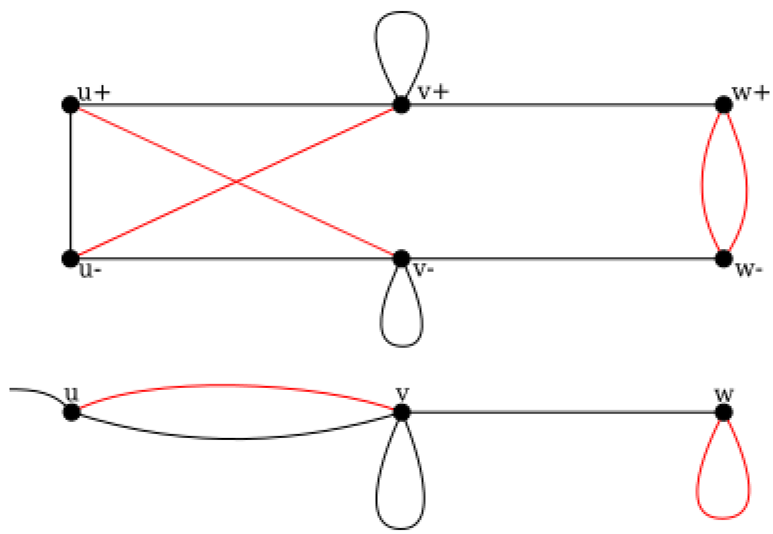
Let Σ be a signed graph. The double cover of Σ is the graph $\widetilde{\Sigma}$ which has

$$V(\widetilde{\Sigma}) = V(\Sigma) \times \{+,-\},$$
$$E(\widetilde{\Sigma}) = E(\Sigma) \times \{+,-\}$$

and projects to Σ via

$$p : \widetilde{\Sigma} \to \Sigma : v^+, v^- \mapsto v, \ \tilde{e}, \tilde{e}^* \mapsto e.$$

We shorten $(v,+)$ and $(v,-)$ to v^+ and v^-. For an edge $e \in E(\Sigma)$, we write $\tilde{e}$ to denote $(e,+) \in E(\widetilde{\Sigma})$ and $\tilde{e}^*$ to denote $(e,-) \in E(\widetilde{\Sigma})$. If $e{:}uv$ in Σ, then $\tilde{e}$ has endpoints $(u,+)$ and $(v,\sigma(e))$ and $\tilde{e}^*$ has endpoints $(u,-)$ and $(v,-\sigma(e))$. Throw out loose edges. A half edge $e{:}v \in E(\Sigma)$ produces a single edge $\tilde{e}$ with endpoints v^+ and v^-. The following diagram illustrates the possibilities in the double cover construction; black edges are positive and red edges are negative. Edges in $\widetilde{\Sigma}$ are taken to be positive; the colors are merely for illustrating the correspondence.



We see there is a degree of symmetry to the double cover. This is captured by the canonical involution $* : \widetilde{\Sigma} \to \widetilde{\Sigma} : v^+ \leftrightarrow v^-, \tilde{e} \leftrightarrow \tilde{e}^*$. Switching v in Σ interchanges v^+ and v^-, pulling edges along with them. This is an automorphism of $\widetilde{\Sigma}$. Regarding $\widetilde{\Sigma}$ as unsigned allows us to consider it as covering the whole switching class of Σ.

If Σ is connected and balanced, then $\widetilde{\Sigma}$ has two connected components. If Σ is connected and unbalanced, then $\widetilde{\Sigma}$ will be connected. The double cover of a graph Γ with all edges negative, $\widetilde{-\Gamma}$, is widely known under the name "bipartite double cover" of Γ.

For an oriented signed graph, an orientation of $\widetilde{\Sigma}$ is determined by the relationship $\tilde{\tau}(v^\varepsilon, e) = \varepsilon\tau(v,e)$. Here ε is either $+$ or $-$.

A surface S has its oriented double cover $\widetilde{S}$. If $\Sigma \hookrightarrow S$ (an orientation embedding), then $\widetilde{\Sigma} \hookrightarrow \widetilde{S}$ (an ordinary embedding). The cover $\widetilde{S}$ comes with a canonical involution $* : \widetilde{S} \to \widetilde{S}$ which restricts to $* : \widetilde{\Sigma} \to \widetilde{\Sigma}$ under the embedding.

Altogether, an orientation embedding of Σ in S is equivalent to an antipodal embedding of $(\widetilde{\Sigma}, *)$ in $(\widetilde{S}, *)$.

Proposition 117. *$(\widetilde{\Sigma}, *)$ can be thought of as a graph Γ with a fixed-vertex-free involutory automorphism * [Editors note: a free action of $\mathbb{Z}/2\mathbb{Z}$ on Γ]. We can give $\Gamma/*$ a signature by making a circle in $\Gamma/*$ positive $\iff$ the preimage of the circle in Γ is two disjoint circles. Then $\Gamma/*$ is switching equivalent to Σ.*

Proof. Trace circles in the quotient $\Gamma/*$. The lift has degree 2 or less at all its vertices so it has to be a circle. The preimage of any vertex has two elements, so the lift has at most two distinct components. □

Proposition 118. *Let $\Delta \subseteq \Sigma$ be connected. If Δ is unbalanced, its preimage in $\widetilde{\Sigma}$ is connected. If Δ is balanced, its preimage in $\widetilde{\Sigma}$ is two disjoint copies of Δ.*

As a summary remark, lots of things work nicely because switching behaves well with respect to double covering of both graphs and surfaces.

Looking for a moment at coloring, a coloration $\kappa : V \to \{-k, \dots, +k\}$ lifts to $\tilde{\kappa} : \widetilde{V} \to \{-k, \dots, +k\}$ by

$$\tilde{\kappa}(v^+) = \kappa(v), \quad \tilde{\kappa}(v^-) = -\kappa(v).$$

Proposition 119. *The improper edge set of κ lifts to the improper edge set of $\tilde{\kappa}$; i.e., $I(\tilde{\kappa}) = p^{-1}(I(\kappa))$. In particular, $\tilde{\kappa}$ is proper if and only if κ is proper.*

Proof. Check it and see ;). □

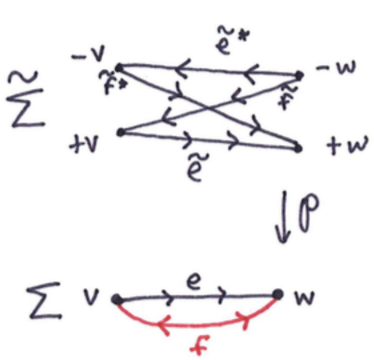
As a corollary, $\tilde{\kappa}$ is proper $\iff$ κ is proper.

Things get complicated when we talk about switching, which is part of the reason we didn't use the double cover for our region problem. [**ADD some discussion.**]

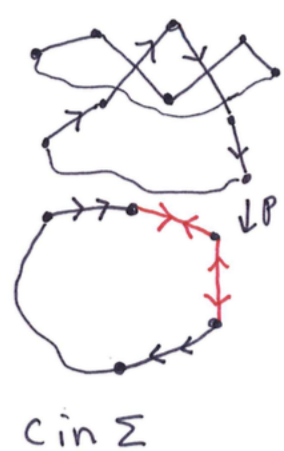
Notes for 8 May 2017 – Amelia Mattern.

We continue our discussion of double covers of signed graphs. In the examples for today, black lines indicate positive edges and red lines indicate negative edges. Also, to keep the discussion simple we assume the graphs have no loose or half edges.

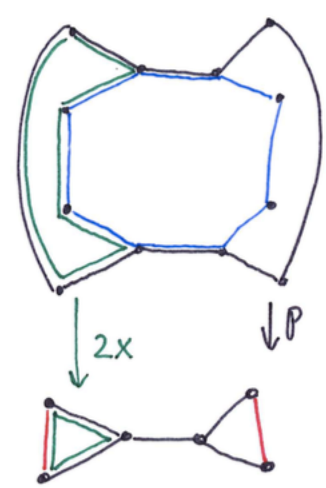
Example 8. A basic example.



Example 9. Here both covers are cycles.



Example 10. Notice that Σ is the image of a circle, and that the green edges above trace the green edges below twice.



12.2. **Integral Flows.**

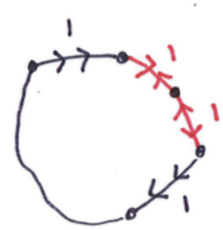
There is a substantial theory of integral flows on graphs and there is beginning to be a similar theory for signed graphs. I can only give some minimal background, as we have but a day.

Frame circuits in Σ are positive circles or contrabalanced handcuffs. They are the minimal dependent sets of a matroid (the *frame matroid*), which I avoid since we don't do matroids in these notes.

An *integral flow* is a function $f : E \to \mathbb{Z}$, defined on oriented edges with the usual convention that reorienting an edge negates (i.e., inverts) the flow value. The set of flows, or *flow space*, is the null space of the incidence matrix H. An integral flow is an integral 1-cycle in the sense of homology; multiplication by the incidence matrix is the boundary mapping..

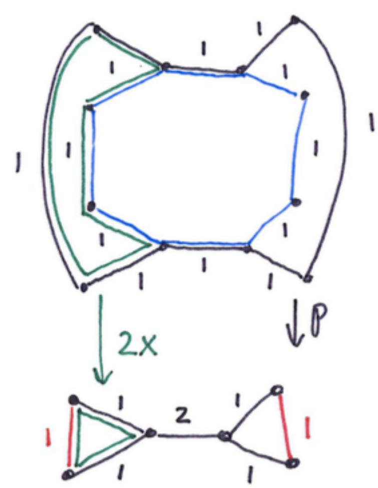
Definition 33. A flow φ_1 *conforms* to φ_2 if $\varphi_2(e)\varphi_1(e) \geq 0$ and $|\varphi_1(e)| \leq |\varphi_2(e)|$ for all edges e. We say φ is *conformally decomposable* if $\varphi = \varphi_1 + \varphi_2$ where φ_1 and φ_2 both conform to φ and neither one is the zero flow.

Example 11. Below is an example of a indecomposable integral flow on the circle from Example 9.



Definition 34. A flow φ is *minimal* (or *irreducible*) if it is conformally indecomposable and is not 0.

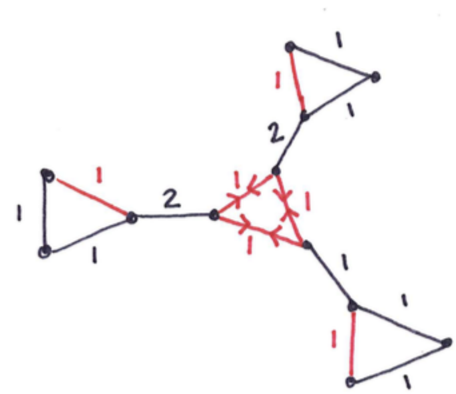
Example 12. Below is an example of a minimal flow on our graph from Example 10, shown both in Σ and in $\tilde{\Sigma}$.



We want to find minimal flows on signed graphs. For instance, a *circuit flow* is a minimal flow on a circuit; it has the value ± 1 on each edge of a circle in the circuit and the value ± 2 on each edge of the connecting path of a loose handcuff.

A main question: Is every integral flow a sum of circuit flows? Yes for graphs. For signed graphs that is not so; there are irreducible integral flows that are not circuit flows.

Example 13. Here is an example of an integral flow that is minimal, but not a circuit flow.



The reason may have to do with the properties of the incidence matrix. A matrix M is *totally unimodular* (t.u.) if every minor (i.e., subdeterminant) is 0 or ± 1. A matrix is *totally dyadic* (t.d.) if every minor is 0 or $\pm 2^k$ for $k \in \mathbb{Z}$. The incidence matrix of a graph is totally unimodular, but that of a signed graph is not, though it is totally dyadic. The importance of t.u. matrices comes from the fact that a matrix M is totally unimodular if and only if for every integral $\vec{b}$, $M\mathbf{x} = \mathbf{b}$ has an integral solution (if it has a solution). (This property of a t.u. matrix follows immediately from Cramer's Rule, since the denominator in a Cramer's Rule solution is a minor of M. The converse follows from suitable choices of $\mathbf{b}$.)

For us the main theorem about total unimodularity is this old result of Heller and Tompkins.

Theorem 120 (Heller and Tompkins [8])**.** *A matrix M in which every column has at most two nonzero entries, which are ± 1, is t.u. if and only if some rows can be negated so that every column with two nonzeros has one $+1$ and one -1.*

Translating into signed-graphic language, an incidence matrix $\mathrm{H}(\Sigma)$ is t.u. if and only if Σ (with half-edges deleted) is balanced.

Notes for 9 May 2017 – Micah Loverro.

For the sake of good notation we always begin by orienting Σ to obtain a definite incidence matrix H (Eta). The oriented signed graph is now a bidirected graph B (Beta). An *integral flow* on B is a function $f : E(\mathrm{B}) \to \mathbb{Z}$ such that $\mathrm{H}f = 0$, i.e., $f \in \mathrm{Nul}(\mathrm{H})$. The *support* of f is $\operatorname{supp} f := E \setminus f^{-1}(0)$. The orientation is merely a notational necessity; we may reorient edges by the rule that reorienting an edge negates the flow value on that edge, and a flow will always be transformed into a flow on the reoriented graph. Consequently, we may always choose an orientation such that $f \geq 0$, if we wish. We regard corresponding flows on different orientations as the same flow, only described with different notation; thus, an integral flow is really defined on Σ, not on any one orientation; but we need the orientation to state the flow values.

Let $f : E(\mathrm{B}) \to \mathbb{Z}$ be any function. Add up the values on edges that are oriented out of v; call that the *outflow* from v. Similarly define the *inflow* to v. Then f is a flow on Σ if and only if every vertex has equal inflow and outflow. We call a vertex *conservative* (for the obvious reason). Conservation at a vertex is preserved by reorienting edges and negating the flow values on those edges.

Let W be a closed walk $v_0 e_1 v_1 \cdots e_l v_l$. We define the *closed walk flow* f_W of W as follows. Initially, let f be the zero flow. Trace the edges in W from beginning to end. At e_1, let $\varepsilon = - \pm \tau(v_0, e_1)$ (either choice is acceptable) and add ε to $f(e_1)$. Thus, we add 1 if e_1 is directed away from v_0 but we subtract 1 if it is directed towards v_0. Now at each new vertex v_i $(i > 0)$, if the vertex is incoherent negate ε, but preserve ε if the vertex is coherent. Then, add ε to $f(e_{i+1})$. Continue in this way until the walk is finished. Then f is the W-flow f_W. (Its negative, obtained by choosing the opposite sign for the initial ε, is an equally good W-flow. We fix one choice of ε to be called f_W just to make the notation definite.) If W is a coherent walk, then choosing the initial $\varepsilon = +$ ensures that $f_W \geq 0$.

Suppose we have two flows, f and f'. We say f' *conforms* to f if $|f'(e)| \leq |f(e)|$ and $f(e)f'(e) \geq 0$ for every edge in Σ. That means $f(e) = 0 \Rightarrow f'(e) = 0$, and if $f(e) \neq 0$, then $f'(e)$ is 0 or has the same sign as $f(e)$.

Suppose we have an arbitrary flow f on B. By reorienting we can make $f \geq 0$. Define $\|f\| := \sum_{e \in E} f(e)$. Find a coherent closed walk W in $\operatorname{supp} f$ such that f_W conforms to f. We are able to do this because flow is nonnegative and is conserved at every vertex, so if we trace a coherent walk from any vertex of $\operatorname{supp} f$ we can never be prevented from leaving a vertex we entered, except at the initial vertex if we return having used up all its inflow. Thus we are always able to end at the starting point of our walk W. Then $f - f_W \geq 0$ and $\|f - f_W\| < \|f\|$. It follows by induction that:

Proposition 121. *Every flow f is a sum of conforming closed walk flows.*

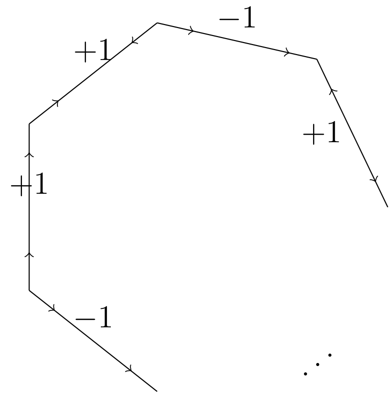
There is a stronger decomposition of flows in graph theory. Here, once again, a graph behaves exactly like an all-positive signed graph. A *circuit flow* on a circle C is a flow f_C; its value is ± 1 on each edge of C and 0 off C. An integral flow f on a graph or signed graph is *irreducible* if it cannot be written as a sum of two nonzero integral flows, both conforming to f.

Theorem 122. *For unsigned graphs, every integral flow is a sum of conforming circuit flows.*

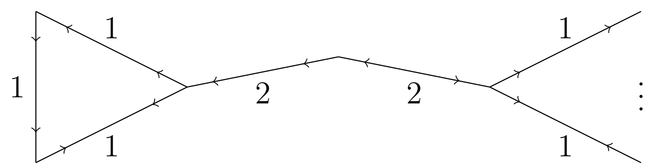
In other words, the only irreducible integral flows on a graph are the circuit flows.

Although the definitions extend readily to signed graphs, this simple decomposition theorem does not. To explain the situation with signed graphs needs quite a bit of discussion, of which I will only give a small part.

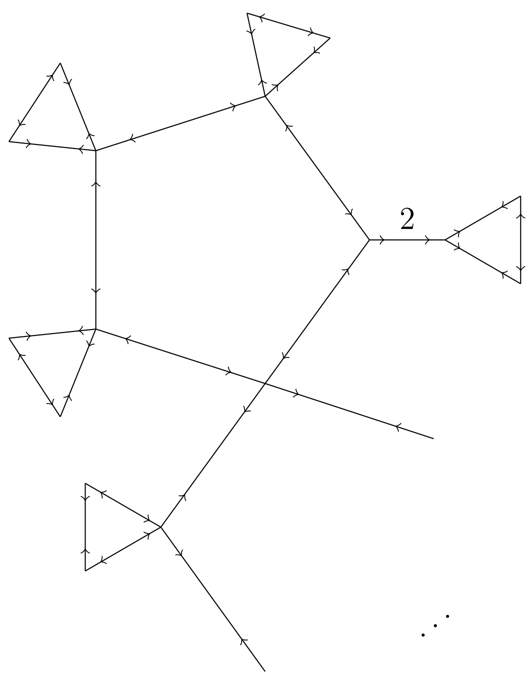
We first take a look at some examples of irreducible flows. Suppose C is a positive circle such as the one below. Edges e are marked with the values of $f_C(e)$. If e is not an edge of the circle, then $f_C(e)$ is defined to be 0. Notice that reversing the orientation of all negative edges yields an orientation which is coherent at each vertex.



Edges can also take flow value more than 1. If C is the graph below, the flow pictured is also irreducible. Here we have already given the graph a coherent orientation.



The previous two examples are circuit flows on a signed graph. On signed graphs, however, irreducible flows can be more complex than circuit flows. In the graph below, the value of $f_C(e)$ is 1 unless labeled. This graph has no source or sink; that is one of the properties of the support of any nonnegative flow.



Chen and Wang discovered a flow decomposition like that for graphs by characterizing all irreducible integral flows on a signed graph. The irreducible flows are too complicated to describe here (under limitations of time), but the picture above gives an idea of what they look like. Our proof in [4], which is more elegant than their original proof (because it depends on general properties instead of case-by-case analysis), depends on lifting to the double covering graph.

Relation to the double covering graph.

If f is a flow on the double covering graph $\widetilde{\Sigma}$ of Σ and p is the covering projection, then for an edge in Σ we define $(pf)(e) = f(\tilde{e}) + f(\tilde{e}^*)$ as the projected flow pf. It is a fact of double covering graphs that any flow of Σ can be lifted to a flow of $\widetilde{\Sigma}$. Flows of irreducible coherent closed walks W_i in Σ lift to circle flows in $\widetilde{\Sigma}$. That let us use the graph decomposition into circuit flows (on $\widetilde{\Sigma}$) to obtain a signed-graph decomposition of flows on Σ. The details were not so simple!

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