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Bound Graphs of a Partially Ordered Set

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In the *upper bound graph* of a set X partially ordered by $<$, the vertex set is X and xy is an edge if there is a $z \in X$ with $x, y \leqslant z$. The *strict upper bound graph* requires $x, y < z$. In the *double bound graph*, xy is an edge if there are w, z such that $w \leqslant x, y \leqslant z$. These graphs are characterized as the graphs having certain kinds of edge coverings by cliques. We observe that the simplest strict upper bound graphs are those of interval orders.

In this note, a *partial order* is an irreflexive, transitive binary relation and a *partially ordered set* (poset) is a non-empty set together with a partial order defined on it. All sets are finite and all graphs are undirected without loops or multiple edges.

There are many graphs that can be defined on a given poset $(X, <)$ having X as vertex set and using $<$ in some way to define the edges. One extensively studied notion, for example, is that of the comparability graph, where xy is an edge if and only if $x < y$ or $y < x$. A characterization of comparability graphs can be found in [4] and [5].

Here we define some new graphs associated with a poset $(X, <)$. The *upper bound graph* (UB-graph) is the graph $U = (X, E_U)$ where $xy \in E_U$ if and only if $x \neq y$ and there exists $m \in X$ such that $x, y \leqslant m$. We say that a graph G is a *UB-graph* if there exists a poset whose upper bound graph is isomorphic to G. The *lower bound graph* (LB-graph) of $(X, <)$ is defined analogously. Clearly a graph is a UB-graph if and only if it is an LB-graph. The *double bound graph* (DB-graph) of $(X, <)$ is the graph $D = (X, E_D)$ where $xy \in E_D$ if and only if $x \neq y$ and there exist $m, n \in X$ such that $n \leqslant x, y \leqslant m$.

Our characterizations will show that every UB-graph is a DB-graph.

However, any path of length greater than two is an example of a DB-graph that is not a UB-graph.

Recall that a *clique* in the graph $G = (V, E)$ is the vertex set of a maximal complete subgraph, and that a family $\mathcal{C}$ of complete subgraphs *edge covers* G if and only if for each $xy \in E$, there exists $C \in \mathcal{C}$ such that $x, y \in C$.

THEOREM 1. *A graph $G = (V, E)$ is a UB-graph if and only if there exists a family $\mathcal{C} = \{C_1, \ldots, C_n\}$ of complete subgraphs of G such that*

(i) *$\mathcal{C}$ edge covers G, and*

(ii) *for each C_i, there is an $x_i \in C_i \setminus (\bigcup_{j \neq i} C_j)$.*

Furthermore, such a family $\mathcal{C}$ must consist of cliques of G and is the only such family.

Proof. Assume G is the UB-graph of the poset $(V, <)$ and let M denote the set of maximal elements in $(V, <)$. For each $m \in M$ set $C(m) = \{x : x \leqslant m\}$. Clearly $C(m) = \{m\}$ if and only if m is an isolated point (both maximal and minimal) of $(V, <)$. It can be routinely verified that the nontrivial $C(m)$'s form a family of complete subgraphs satisfying conditions (i) and (ii).

Conversely, suppose $\mathcal{C}$ is a family of complete subgraphs of G satisfying (i) and (ii). For each i, select a fixed element $x_i \in C_i$ as guaranteed by condition (ii). Define the partial order $<$ on V by $x < x_i$ if and only if $x \in C_i$ $(x \neq x_i)$, with no other elements of V comparable with respect to $<$. Then $(V, <)$ is a "bipartite" poset that consists of only maximal and minimal elements and whose UB-graph is G. Each $C_i \in \mathcal{C}$ is maximal, since $yx_i \in E$ implies $\{y, x_i\} \subseteq C_j$ for some j and then $x_i \in C_j$ implies $i = j$. The uniqueness of $\mathcal{C}$ is observed in [1]. ∎

Clearly the above construction of the poset $(V, <)$ is not unique: quite different posets can have the same UB-graph. However one can characterize those UB-graphs that arise from a unique poset [6].

Let $k(G)$ be the least number of cliques required to edge cover G and let $l(G)$ be the number of cliques having a vertex not in any other clique of G.

THEOREM 2. *Any graph G is an induced subgraph of a UB-graph H. The least number of vertices in H not in G is $k(G) - l(G)$.*

Proof. Let $\mathcal{K} = \{K_1, \ldots, K_m\}$ be an edge covering family of cliques of G. We can construct H by adjoining new vertices k_i to G and edges $k_i x$ for $x \in K_i$, for all $i = 1, 2, \ldots, m$. We need not add k_i if K_i already contains a vertex x_i not in any of the other cliques in $\mathcal{K}$; otherwise we must add k_i. Any clique of G, say K, having a vertex in no other clique of G, must be in $\mathcal{K}$, so we can omit at least $l(G)$ of the k_i's. On the other hand suppose we can omit k_i. This means that K_i has a vertex x_i in no other clique of $\mathcal{K}$. Then K_i consists of x_i and all vertices adjacent to x_i. It follows that x_i lies in no clique of G other than K_i, so K_i is counted in $l(G)$. Thus we cannot omit more than $l(G)$ of the m vertices k_i. Since the minimum m equals $k(G)$, the theorem is proved. ∎

It follows that a forbidden subgraph characterization of UB-graphs is impossible. However one can say that the UB-graphs for which every induced subgraph is a UB-graph are precisely the comparability graphs of trees (noted in [2] and [7]).

Choudom, Parthasarathy and Ravindra [2] (also see Brigham and Dutton [1]) characterized graphs G for which the vertex clique cover number equals $k(G)$ as graphs that satisfy conditions (i) and (ii) of Theorem 1. Tying this in with posets gives us another view of their work. Also, we can place the consanguinity graph of Florence [3] in this framework by observing that a graph G is a consanguinity graph if and only if G is the consanguinity graph of the digraph of some poset. G is the consanguinity graph of the digraph of a poset P if and only if G is the UB-graph of P.

THEOREM 3. *A graph $G = (V, E)$ is a DB-graph if and only if there exist a family of cliques $\mathcal{C} = \{C_1, \ldots, C_n\}$ and disjoint independent subsets M and N of V such that*

(i) *$\mathcal{C}$ edge covers G, and*

(ii) *for each C_i there exist $x_i \in M$, $y_i \in N$ such that*

$$\{x_i, y_i\} \subseteq C_i \text{ and } \{x_i, y_i\} \nsubseteq C_j \text{ for any } j \neq i.$$

Furthermore $\mathcal{C}$ is the unique, minimal edge covering family of cliques in G.

Proof. Assume G is the DB-graph of the poset $(V, <)$. Let M be the set of non-isolated maximal elements and N the set of non-isolated minimal elements. For each $x \in M$ and $y \in N$ with $y < x$, let $C(x, y) = \{z \in V: y \leqslant z \leqslant x\}$. Then the $C(x, y)$'s are a family of cliques of G satisfying (i) and (ii).

Suppose the sets M, N and the family of cliques $\mathcal{C}$ satisfy conditions (i) and (ii). For each $C_i \in \mathcal{C}$, let $\{x_i, y_i\}$ be a fixed set given by (ii). Define the poset $(V, <)$ by setting $y_i < x_i$ and $y_i < z < x_i$ for each $z \in C_i \setminus \{x_i, y_i\}$, with no other comparabilities. Then G is the DB-graph of $(V, <)$.

Given a particular M and N, $\mathcal{C}$ is determined since there is one $C_i \in \mathcal{C}$ for each edge x_iy_i with $x_i \in M$, $y_i \in N$, and $C_i = \{x_i, y_i\} \cup \{z \in V: zx_i, zy_i \in E\}$. From this we have that C_i is the only clique of G containing x_iy_i. Hence any edge covering family of cliques contains $\mathcal{C}$, which is therefore the unique minimal such family. ∎

Theorem 3 shows that every bipartite graph is a DB-graph. The example $G = K_3$ shows that M and N need not be uniquely determined.

For a variation on the above theme define the *strict UB-graph* of a poset $(X, <)$ to be the graph (X, E) where $xy \in E$, $x \neq y$, if and only if there exists $m \in X$ such that $x, y < m$. The next theorem can also be proved easily. In what follows, $\bar{K}_n$ will denote the graph with n vertices and no edges.

THEOREM 4. *The graph $G = (V, E)$ is a strict UB-graph if and only if there exists a family $\mathcal{C} = \{C_1, \ldots, C_m\}$ of cliques that edge covers G and $V = C_1 \cup \ldots \cup C_m \cup \bar{K}_n$ for some $n \geqslant m$, where $\bar{K}_n$ has no vertices in common with any C_i.*

An interesting problem is to determine the UB or strict UB-graph of various classes of posets. For example, what kinds of posets have the simplest UB or strict UB-graphs (that is, UB-graphs of the form K_m or strict UB-graphs of the form $K_m \cup \bar{K}_n$)? The following two statements are clear.

The UB-graph of the poset $(X, <)$ is K_m for some m if and only if $(X, <)$ has a unique maximal element.

The graph $G = (V, E)$ is the UB-graph of a totally ordered set if and only if $G = K_m$ for some m. (Of course posets that are not totally ordered can, by the previous statement, have K_m as their UB-graphs.)

Recall that an *interval order* is a poset $(X, <)$ where $x < y$ and $z < w$ imply that $x < w$ or $z < y$.

THEOREM 5. *The graph $G = (V, E)$ is the strict UB-graph of an interval order if and only if $G = K_m \cup \bar{K}_n$ for some m and n.*

Proof. Assume G is the strict UB-graph of an interval order $(V, <)$. For each $x \in V$, let $H^*(x) = \{z \in V : x < z\}$. Then $xy \in E$ if and only if $H^*(x) \cap H^*(y) \neq \emptyset$. Rabinovitch [8] has shown that for each $x, y \in V$, $H^*(x) \subseteq H^*(y)$ or $H^*(y) \subseteq H^*(x)$. So for non-maximal elements $x, y \in V$, $H^*(x) \cap H^*(y) \neq \emptyset$ and hence $G = K_m \cup \bar{K}_n$.

Conversely, if $G = K_m \cup \bar{K}_n$, form the poset $(V, <)$ by totally ordering the m elements in K_m and setting the n elements of $\bar{K}_n$ as the maximal elements, so $x < y$ for all $x \in K_m$ and $y \in \bar{K}_n$, Then $(V, <)$ is an interval order whose strict UB-graph is G. ∎

We caution the reader that there are non-interval orders having strict UB-graphs isomorphic to $K_m \cup \bar{K}_n$.

Additional properties of UB-graphs will appear in [7].

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