PROPOSITION 4.4. *Let Ω be a finite biased graph that is full and complete. Let M be an intermediate matroid on E; that is, $L(\Omega) \leqslant M \leqslant G(\Omega)$. Then $M = L(\Omega)$ or $G(\Omega)$.*

Proof. Let Ω be full and complete; we may assume there is one unbalanced loop h_i at each node v_i and that all other edges are links. Let M be an intermediate matroid, $L = L(\Omega)$, and $G = G(\Omega)$. Let $H_{ij} = \{h_i, h_j\}$ and $C(e_{ij}) = H_{ij} \cup e_{ij}$, where e_{ij} denotes a link $v_i v_j$.

We show that either all H_{ij} are circuits or all $C(e_{ij})$ are. Suppose H_{ij} and H_{jk} are circuits; then by circuit exchange (and since $M \geqslant L$) H_{ik} is a circuit. Suppose $C(e_{ij})$ and $C(e_{jk})$ are circuits. By circuit exchange there is a circuit $D \subseteq \{h_i, e_{ij}, e_{jk}, h_k\}$ containing e_{ij}. Since D contains a lift circuit, which can only be H_{ik}, H_{ik} itself is not a circuit in M. Therefore $C(e_{ik})$ is a circuit for each link between v_i and v_k.

We show next that each circuit C of M is a bias or lift circuit. Suppose a circuit C is neither a bias nor a lift circuit. Then C contains no bias circuit, so its components are r unbalanced unicycles $U_1, U_2, \ldots, U_r$ and perhaps some trees, where $r \geqslant 2$ because $M \geqslant L$. Let C_i be the circle in U_i, $v_i \in N(C_i)$, and $e_1 \in C_1$ and let $e: v_1 v_2$ be a link. $C_1 \cup C_2$ is independent in M, for otherwise C would be a lift circuit. The set $D = C_1 \cup C_2 \cup e$, being a bias circuit, is dependent in M, hence a circuit. By circuit exchange between C and D, $(C \cup e) \backslash e_1$ contains a circuit C', whose cyclomatic number is necessarily lower than that of C. Since C' cannot be a lift circuit (because if $C' \not\subset C$, it contains the isthmus e) or a bias circuit, it in turn can be modified as above. Eventually one gets a circuit with at most one circle, but that contradicts $M \geqslant L$. Hence after all C must have been a bias or lift circuit.

Suppose now that M has a circuit C which is a lift circuit but not a bias circuit; that is, $C = C_1 \cup C_2$, where C_1 and C_2 are node-disjoint unbalanced circles. Let $v_i \in N(C_i)$ and $e_1 \in C_1$. By exchange with the circuit $C_1 \cup h_1$, there is a circuit $C' \subseteq C_1 \cup C_2 \cup h_1 \backslash e_1$. C' can only be $C_2 \cup h_1$. Exchange with $C_2 \cup h_2$ leads to the conclusion that H_{12} and consequently all H_{ij} are circuits.

On the other hand suppose M has a circuit C which is a bias but not a lift circuit. Thus C is a loose handcuff $C_1 \cup C_2 \cup P$, P being a path connecting $v_1 \in N(C_1)$ to $v_2 \in N(C_2)$. By circuit exchange with $C_1 \cup h_1$, there is a bias or lift circuit $C' \subseteq C \cup h_1 \backslash e_1$ (where $e_1 \in C_1$), which can only be the bias circuit $C_2 \cup P \cup h_1$ or the lift circuit $C_2 \cup h_1$. Actually the former obtains, for if $C_2 \cup h_1$ were a circuit, exchange with $C_1 \cup h_1$ would imply that $C_1 \cup C_2$ is dependent, contradicting C's being a circuit. By exchanging with $C_2 \cup h_2$, we deduce that $P \cup h_1 \cup h_2$ is a circuit. Thus H_{12} is not; it follows that all $C(e_{ij})$ are circuits.

From the last two paragraphs we conclude that M equals L or G. ∎

It seems that Proposition 4.4 can be generalized to full biased graphs which are 2-connected (ignoring nonlink edges) and that if Ω is full but not 2-connected one can characterize all intermediate matroids. These results may appear elsewhere. At any rate Proposition 4.4 suffices to prove the main result.

THEOREM 4.5. *The only intermediate-matroid constructions with domain all biased graphs are* **G** *and* **L**.

Proof. This follows from the negative answer of Proposition 4.4 to Problem 4.3(a) in the case of full, complete biased graphs, provided we show that every biased graph is a subgraph of a complete biased graph.

Let Ω be a given biased graph and Γ its complementary graph, in which nodes are adjacent precisely when they are not adjacent in Ω. The biased union $\Omega \sqcup [\Gamma]$ is a complete graph containing Ω, as desired. ∎

The generalization of Proposition 4.4 mentioned earlier suggests that little of interest will be found in answer to Problem 4.3(b) unless full biased graphs are ruled out of the construction domain. I believe there may exist intermediate constructions other than **G** and **L** with domain all biased graphs having no unbalanced edges, but I do not have any examples.

5. INFINITARY ANALOGS

Klee [13] and Bean and Higgs [1, 11] have noted the existence of infinitary matroids analogous to the usual (finitary) bicircular and graphic matroids $G(\Gamma, \varnothing)$ and $G(\Gamma)$ of an infinite graph Γ. Each is defined by adding to the circuits of the finitary version infinite circuits based on *rays* (one-way infinite paths) or *beams* (two-way infinite paths). Klee's infinitary bicircular matroid has two kinds of infinite circuits: a beam, and the union of a ray and an unbalanced figure which meet only at the ray's initial node. In the infinitary graphic matroid $G^{\infty}(\Gamma)$ (due to Bean [1, 11] and independently due to Klee) the only infinite circuits are beams. In accordance with the guiding principle (which, to be sure, must be applied selectively) that any property of a graph should generalize to biased graphs, we should expect to have similar infinitary bias and lift matroids by adding appropriate infinite circuits.

Before approaching these problems we should recall some terminology of infinite matroids from [13]. (Bean and Higgs' terminology differs but they also take an operator approach to infinite matroids.) Klee defines the following properties of an enlarging, isotone operator f on subsets of a set E. We let X, Y denote subsets of E.

(I) $f^2(Y) = f(Y)$. (Idempotence.)

(E) If $e \in f(Y)$ and $e \notin f(Y \backslash X)$, then $x \in f((Y \cup e) \backslash x)$ for some $x \in X$. (Exchange.)

(C) If $e \in f(Y)$, there is a minimal $U \subseteq Y$ for which $e \in f(U)$ and U is independent. (Weak circuit closure.)

(U is *independent* if $u \notin f(U \backslash u)$ for every $u \in U$.) Let $\mathscr{C}$ be a *clutter*, a class of subsets of E of which none contains another. Let the *$\mathscr{C}$-closure* of $S \subseteq E$ be

$$\operatorname{clos}(S, \mathscr{C}) = S \cup \{e \in E\colon \text{there is a } C \in \mathscr{C} \text{ such that } e \in C \subseteq S \cup e\}.$$

Klee observes that $\operatorname{clos}(\cdot, \mathscr{C})$ is an enlarging, isotone operator satisfying (E) and (C). He defines an IEC-*matroid* (a special kind of matroid; Klee's "matroids" are very general) to be an enlarging, isotone operator satisfying (I), (E), and (C).

We define the *infinitary bias matroid* $G^\infty(\Omega)$ by its circuit class $\mathscr{C}_G^\infty(\Omega) = \{S \subseteq E\colon S$ is a bias circuit, a beam, or a union of an unbalanced figure and a ray meeting only at the starting point of the ray$\}$. This is the class obtained from the definition of bias circuit (Section 2) modified by declaring a ray to be an unbalanced figure. We define the operator $\operatorname{clos}_G^\infty$ to be $\operatorname{clos}(\cdot, \mathscr{C}_G^\infty(\Omega))$, leaving Ω implicit. We call a component of $S \subseteq E$ (that is, of the subgraph (N, S)) *long* if it contains a one-way infinite path, *short* if it does not. Let $N_\infty(S)$ be the union of node sets of long components of S. We also need to state the *strong circuit exchange* property of a clutter $\mathscr{C}$:

($\tilde{\gamma}$) If C_1 and C_2 are in $\mathscr{C}$, $e \in C_1 \cap C_2$, and $f \in C_1 \backslash C_2$, then there exists $C \in \mathscr{C}$ such that $f \in C \subseteq (C_1 \cup C_2) \backslash e$.

THEOREM 5.1. *Let Ω be a biased graph. The operator $\operatorname{clos}_G^\infty$ defines an* IEC-*matroid whose circuits are the members of $\mathscr{C}_G^\infty(\Omega)$. We have*

$$\operatorname{clos}_G^\infty S = E{:}[N_0(S) \cup N_\infty(S)] \cup \bigcup_{\substack{B \in \pi_b(S) \\ \text{short}}} \operatorname{bcl}(S{:}B).$$

Furthermore, $\mathscr{C}_G^\infty(\Omega)$ has the strong circuit exchange property.

Proof. The main step is to show that $\operatorname{clos}_G^\infty$ satisfies the stated expression. Clearly $\operatorname{clos}_G(S) \subseteq K \equiv \operatorname{clos}_G^\infty(S)$. Thus $\operatorname{bcl}(S{:}B) \subseteq K$ if $S{:}B$ is a balanced component and $E{:}B \subseteq K$ if $S{:}B$ is an unbalanced component. If $S{:}B$ is a long component, any edge in $S^c{:}B$ forms with S a circle touching some ray in $S{:}B$, which gives either a balanced circle or a circle-and-ray circuit on e in $S \cup \{e\}$. Thus $E{:}B \subseteq K$. Now consider an edge bridging two components of S. If either one is short and balanced, clearly $e \notin K$. In the remaining cases there is a contrabalanced handcuff, an unbalanced-figure-and-ray, or a beam on e; thus $e \in K$.

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(E) If $e \in f(Y)$ and $e \notin f(Y \backslash X)$, then $x \in f((Y \cup e) \backslash x)$ for some $x \in X$. (Exchange.)

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Klee observes that $\operatorname{clos}(\cdot, \mathscr{C})$ is an enlarging, isotone operator satisfying (E) and (C). He defines an IEC-*matroid* (a special kind of matroid; Klee's "matroids" are very general) to be an enlarging, isotone operator satisfying (I), (E), and (C).

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THEOREM 5.1. *Let Ω be a biased graph. The operator $\operatorname{clos}_G^\infty$ defines an* IEC-*matroid whose circuits are the members of $\mathscr{C}_G^\infty(\Omega)$. We have*

$$\operatorname{clos}_G^\infty S = E{:}[N_0(S) \cup N_\infty(S)] \cup \bigcup_{\substack{B \in \pi_b(S) \\ \text{short}}} \operatorname{bcl}(S{:}B).$$

Furthermore, $\mathscr{C}_G^\infty(\Omega)$ has the strong circuit exchange property.

Proof. The main step is to show that $\operatorname{clos}_G^\infty$ satisfies the stated expression. Clearly $\operatorname{clos}_G(S) \subseteq K \equiv \operatorname{clos}_G^\infty(S)$. Thus $\operatorname{bcl}(S{:}B) \subseteq K$ if $S{:}B$ is a balanced component and $E{:}B \subseteq K$ if $S{:}B$ is an unbalanced component. If $S{:}B$ is a long component, any edge in $S^c{:}B$ forms with S a circle touching some ray in $S{:}B$, which gives either a balanced circle or a circle-and-ray circuit on e in $S \cup \{e\}$. Thus $E{:}B \subseteq K$. Now consider an edge bridging two components of S. If either one is short and balanced, clearly $e \notin K$. In the remaining cases there is a contrabalanced handcuff, an unbalanced-figure-and-ray, or a beam on e; thus $e \in K$.

Now idempotence is proved. Klee proved that the circuits of $\operatorname{clos}(\cdot, \mathscr{C})$ are the members of $\mathscr{C}$. He also proved that the circuits of an IEC-matroid have strong circuit exchange [13, p. 143, Theorem 5]. ∎

Note that $G^\infty([\Gamma]) = G^\infty(\Gamma)$; so we have a bias-matroid generalization of Klee's and Bean's matroids.

How to find a lift-type generalization is still a mystery. One could define a lift analog of $G^\infty(\Gamma)$ by taking the circuit class $\mathscr{C}$ to be {lift circuits and beams}. But the closure $\operatorname{clos}(\cdot, \mathscr{C})$ is not idempotent, unlike that of $G^\infty(\Gamma)$. Consider the biased graph Ω which has two components, one an unbalanced loop and the other a beam with one edge doubled to form an unbalanced digon. Let S be a beam. Then $T = \operatorname{clos}(S, \mathscr{C})$ is the beam with the doubled edge added and $\operatorname{clos}(T, \mathscr{C}) = E$. This shows that $\operatorname{clos}(\cdot, \mathscr{C})$ disobeys Klee's law (vwI) of very weak idempotence and therefore is not a "matroid" even by his minimum definition. I do not think this is satisfactory for a supposed infinitary lift matroid, especially since there is an idempotent infinitary bias matroid.

One might try to define infinitary lift circuits, by analogy with $\mathscr{C}_G^\infty$, by declaring a ray to be an unbalanced figure. Unfortunately, since (infinitary) lift circuits need not be connected, a disjoint pair of rays would then form a nonminimal "circuit," or perhaps we should say, a dependent set containing no minimal dependent set. This seems worse than the previous example, and certainly does not generalize $G^\infty(\Gamma)$.

Problem 5.2. Find a satisfactory infinitary analog of the lift matroid of a biased graph.

One might try to generalize $G(\Omega)$ differently, by merely adding beams to the list of bias circuits. This, however, does not yield a matroid. Consider a beam B with two unbalanced loops at adjacent nodes. The loops and connecting edge e form a handcuff circuit C. Then $(B \cup C) \backslash e$ contains no beam or bias circuit. Thus even weak circuit exchange fails, which is not very satisfactory in itself and also shows (by [13, Theorem 4]) that the corresponding closure disobeys Klee's (vwI).

We conclude by suggesting that Matthews and Oxley's proof that $G^\infty(\Gamma, \varnothing)$ has bases (that is, is a B-matroid; see [19] for definitions and proof) should generalize to biased graphs.

Problem 5.3. Prove that $G^\infty(\Omega)$ is a B-matroid for any biased graph Ω.

6. Seven Dwarves: Matroids of the Biased K_4's

Our analysis of the seven biased graphs based on K_4, $\Omega_i = \Omega_i(K_4)$ for $i = 1, 2, \ldots, 7$, continues from Section I.7 with descriptions of their matroids and their linear representability. This is a relatively simple matter, for since no two circles in K_4 are node disjoint, $G(\Omega_i)=L(\Omega_i)$ for every $\Omega_i(K_4)$.

From the end of the proof of Theorem 3.1 and the succeeding comment we see that the matroids $L(\Omega_i)^{\perp}$ are precisely the seven nonisomorphic rank-3 one-point extensions of $G(K_4)^{\perp}$—thus of $G(K_4)$, since the latter is self-dual by the edge permutation $v_1v_2 \leftrightarrow v_3v_4$, $v_1v_3 \leftrightarrow v_2v_4$, $v_1v_4 \leftrightarrow v_2v_3$. The lift matroids $L(\Omega_i)$ are dual to these extensions contracted by e_0. Since the duals have lower rank than the lift and complete lift themselves (whose rank is 4 except for $L(\Omega_1)$), it is simpler to describe the duals. That we do below. We also describe the lift matroids for use in Part III and mention special features of some examples. We omit the proofs, which are easy.

Besides describing the matroids we also list in Table 6.1, for comparison with Conjecture 3.14, the fields over which they have linear representations. We also mention where they have projectively unique representation, meaning that any two projective representations are related by a projective transformation. The proofs are easy, given the facts that a matroid M is K-representable if and only if its dual is and that the representation is projectively unique if and only if that is true for the dual. (The reason is that if the matrix (I, A) represents M by column vectors, then $(-A^T, I)$ represents $M^{\perp}$.) Hence in the examples one can work with $L_0(\Omega_i)^{\perp}$ and $L(\Omega_i)^{\perp}$.

Comparing Table 6.1 with Table I.7.1 in light of the first sentence following Conjecture 3.14, we see that the conjecture is valid for any biased K_4. Example 6.4 illustrates exception (iii).

TABLE 6.1.

The Fields K over which $L(\Omega_i(K_4))$ and $L_0(\Omega_i(K_4))$ are Linearly Representable and over which the Representation is Projectively Unique

	Example $\Omega_i=\Omega_i(K_4)$						
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7
Representation of $L(\Omega_i)$							
exists:	all	all	ord $\geqslant 3$	all	ord $\geqslant 3$	ord $\geqslant 4$	ord $\geqslant 5$
is unique:	all	all	ord $= 3$	all	ord $= 3$	none	none
Representation of $L_0(\Omega_i)$							
exists:	all	all	ord $\geqslant 3$	char $= 2$	char $\neq 2$	ord $\geqslant 4$	ord $\geqslant 5$
is unique:	all	all	ord $= 3$	char $= 2$	char $\neq 2$	ord $= 4$	none

Note. ord means the order $|K|$; char means the characteristic of K.

and their linear representability. This is a relatively simple matter, for since no two circles in K_4 are node disjoint, $G(\Omega_i)=L(\Omega_i)$ for every $\Omega_i(K_4)$.

From the end of the proof of Theorem 3.1 and the succeeding comment we see that the matroids $L(\Omega_i)^\perp$ are precisely the seven nonisomorphic rank-3 one-point extensions of $G(K_4)^\perp$—thus of $G(K_4)$, since the latter is self-dual by the edge permutation $v_1v_2 \leftrightarrow v_3v_4$, $v_1v_3 \leftrightarrow v_2v_4$, $v_1v_4 \leftrightarrow v_2v_3$. The lift matroids $L(\Omega_i)$ are dual to these extensions contracted by e_0. Since the duals have lower rank than the lift and complete lift themselves (whose rank is 4 except for $L(\Omega_1)$), it is simpler to describe the duals. That we do below. We also describe the lift matroids for use in Part III and mention special features of some examples. We omit the proofs, which are easy.

Besides describing the matroids we also list in Table 6.1, for comparison with Conjecture 3.14, the fields over which they have linear representations. We also mention where they have projectively unique representation, meaning that any two projective representations are related by a projective transformation. The proofs are easy, given the facts that a matroid M is K-representable if and only if its dual is and that the representation is projectively unique if and only if that is true for the dual. (The reason is that if the matrix (I, A) represents M by column vectors, then $(-A^T, I)$ represents $M^\perp$.) Hence in the examples one can work with $L_0(\Omega_i)^\perp$ and $L(\Omega_i)^\perp$.

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	Example $\Omega_i=\Omega_i(K_4)$						
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7
Representation of $L(\Omega_i)$							
exists:	all	all	ord $\geqslant 3$	all	ord $\geqslant 3$	ord $\geqslant 4$	ord $\geqslant 5$
is unique:	all	all	ord $=3$	all	ord $=3$	none	none
Representation of $L_0(\Omega_i)$							
exists:	all	all	ord $\geqslant 3$	char $=2$	char $\neq 2$	ord $\geqslant 4$	ord $\geqslant 5$
is unique:	all	all	ord $=3$	char $=2$	char $\neq 2$	ord $=4$	none

Note. ord means the order $|K|$; char means the characteristic of K.

EXAMPLE 6.1. Here $L(\Omega_1)=G(K_4)$ and $L_0(\Omega_1)=G(K_4)\oplus(e_0)_1=G(K_4 \cup K_2)$, hence is graphic. (The symbol $\cup$ denotes disjoint union.) Dually, $L_0(\Omega_1)^\perp$ consists of $G(K_4)$ with e_0 added as a loop. $L(\Omega_1)^\perp$ is $G(K_4)$.

EXAMPLE 6.2. In the complete lift e_0 is in series with the exceptional edge e. Consequently $L_0(\Omega_2)=G(\Gamma_0)$, where Γ_0 is obtained from K_4 by subdividing e into two edges called e and e_0. Thus $L_0(\Omega_2)$ is graphic. Deleting e_0 gives $L(\Omega_2)$. Dually, $L_0(\Omega_2)^\perp=G(\Gamma_0^*)$, where Γ_0^* is K_4 with e_0 added in parallel to e. $L(\Omega_2)^\perp$ is a 3-point line $U_{2,3}$ with two points doubled in parallel and with one loop.

EXAMPLE 6.3. In the dual description, $L_0(\Omega_3)^\perp$ is $G(K_4)^\perp$ with e_0 added to one existing 3-point line, namely the complement of the single balanced triangle. $L(\Omega_3)^\perp$ is a 4-point line $U_{2,4}$ with one point tripled in parallel. In $L(\Omega_3)$ there is one dependent line, namely the balanced triangle. The whole matroid consists of this line and the three remaining edges in general position in rank 4.

EXAMPLE 6.4. The dual $L_0(\Omega_4)^\perp$ is the Fano matroid. $L(\Omega_4)^\perp$ is $U_{2,3}$ with every point doubled in parallel. In $L(\Omega_4)$ the only nonspanning dependent flats are the three balanced quadrilaterals, which are planes.

EXAMPLE 6.5. $L_0(\Omega_5)^\perp$ is the non-Fano matroid. To be specific, it is the Fano matroid with one 3-point line on e_0 eliminated in favor of three 2-point lines. The eliminated line consists of e_0 and the complement of the one unbalanced quadrilateral. $L(\Omega_5)^\perp$ is $U_{2,4}$ with two points doubled in parallel. $L(\Omega_5)$ has two nonspanning dependent flats: each balanced quadrilateral is a plane.

EXAMPLE 6.6. $L_0(\Omega_6)^\perp$ is $G(K_4)^\perp$ with e_0 added to one existing 2-point line but otherwise in general position. The chosen 2-point line consists of the two edges not contained in the sole balanced quadrilateral. $L(\Omega_6)^\perp$ is $U_{2,5}$ with one point doubled in parallel. $L(\Omega_6)$ has one nonspanning dependent flat, a plane consisting of the sole balanced quadrilateral.

EXAMPLE 6.7. Since $\Omega_7=(K_4, \varnothing)$ is contrabalanced, $G(\Omega_7)$ $(=L(\Omega_7))$ is the bicircular matroid of K_4. This matroid of Γ is transversal, presented transversally by treating each edge as a subset of $N(K_4)$. Dually, $L_0(\Omega_7)^\perp$ is $G(K_4)^\perp$ with e_0 added in general position in the same plane. $L(\Omega_7)^\perp$ is $U_{2,6}$, so $L(\Omega_7)=U_{4,6}$.

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