PROBLEMS AND SOLUTIONS

Edited by Gerald A. Edgar, Doug Hensley, Douglas B. West
with the collaboration of Paul T. Bateman, Mario Benedicty, Itshak Borosh, Paul
Bracken, Ezra A. Brown, Randall Dougherty, Dennis Eichorn, Tamás Erdélyi, Kevin
Ford, Zachary Franco, Christian Friesen, Ira M. Gessel, Jerrold R. Griggs, Jerrold
Grossman, Kiran S. Kedlaya, Andre Kündgen, Frederick W. Luttmann, Vania Mascioni,
Frank B. Miles, Richard Pfeifer, Cecil C. Rousseau, Leonard Smiley, John Henry Steel-
man, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles
Vanden Eynden, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY
problems address on the inside front cover. Submitted solutions should arrive at
that address before December 31, 2004. Additional information, such as gen-
eralizations and references, is welcome. The problem number and the solver's
name and address should appear on each solution. An acknowledgement will be
sent only if a mailing label is provided. An asterisk (*) after the number of a
problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

11096. Proposed by Said Amghibeck, Quebec, Canada. Show that for each positive
integer n there exists a polynomial \( P_n \) in \( \mathbb{C}[x_1, \ldots, x_n] \) such that, for every \( n \times n \)
matrix \( A \) over \( \mathbb{C} \), \( \det A = P_n[\text{Tr} A, \text{Tr} A^2, \ldots, \text{Tr} A^n] \). (For example, \( x^3/6 - xy/2 + 
z/3 \) will serve for \( P_3 \).)

11097. Proposed by Kevin Ford, University of Illinois, Urbana, IL. For any integer
\( n > 1 \), let \( \omega(n) \) denote the number of distinct prime factors of \( n \), and \( d(n) \) the number
of divisors. For \( 1 \leq i \leq \omega(n) \) let \( p_i \) be the \( i \)th smallest prime factor of \( n \), and for
\( 1 \leq i \leq d(n) \) let \( d_i \) be the \( i \)th smallest positive divisor of \( n \). Define \( v(r) \) by \( 2^{v(r)} \mid r \)
and \( 2^{1+v(r)} \nmid r \).

(a) Let \( n \) be a product of \( k \) distinct primes. For \( 1 \leq j \leq 2^k - 1 \), let \( t = v(j) + 1 \). Prove
that \( d_{j+1}/d_j \leq p_i \).

(b) Generalize (a) to the case of an arbitrary positive integer \( n \).

11098. Proposed by Christopher Hillar and Darren Rhea, University of California,
Berkeley, CA. Let

\[
    f(n) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{2^i - 1} \binom{n}{i}.
\]

Prove that there are constants \( c \) and \( c' \) such that \( c \leq f(n)/\log n \leq c' \) for sufficiently
large \( n \) (that is, \( f(n) = \Theta(\log n) \)).

11099. Proposed by Matthias Beck, San Francisco State University, San Francisco,
CA, Richard Ehrenborg, University of Kentucky, Lexington, KY, and Thomas Za-
slavsky, State Univ. of New York, Binghamton, NY. A \( 3 \times 3 \) square array \( Q \) of nine
distinct integers is semimagic if all the row and column sums are equal, and it is magic
if, in addition, the two diagonals have the same sum as the rows and columns. We
make a set of three-sided row dice from such a square as follows: the sides of die \( i \) are
labelled with the numbers in row $i$. We say that die $i$ beats die $j$ if we expect die $i$ to show a larger number than die $j$ more than half the time.  
(a) Prove that for every $3 \times 3$ magic square each row die beats exactly one other.  
(b) Prove that the same holds for every $3 \times 3$ semimagic square with entries $1, \ldots, 9$.  
(c) Find a $3 \times 3$ semimagic square not satisfying the conclusion of parts (a) and (b).

11100. Proposed by Călin Popescu, Romanian Academy Institute of Mathematics, Bucharest, Romania. Given positive integers $m, n$, and $p$ satisfying $m \leq n \leq p$ and positive real numbers $\alpha, x_1, \ldots, x_p$, prove that

$$n^\alpha \left( \begin{array}{c} p \\ m \end{array} \right) \sum_{I \subseteq \{1, \ldots, p\}, |I| = n} \left( \sum_{i \in I} x_i \right)^{\alpha m} \leq \left( \begin{array}{c} p \\ n \end{array} \right) \sum_{I \subseteq \{1, \ldots, p\}, |I| = m} \prod_{i \in I} x_i^{-\alpha}.$$

Show that

$$\int_0^\infty \frac{a}{\sqrt{a^2 + x^2}} \tan^{-1} \left( \frac{b}{\sqrt{a^2 + x^2}} \right) \, dx = \frac{a \pi}{2} \left[ \log \left( b + \sqrt{b^2 + a^2} \right) - \log a \right].$$

11102. Proposed by Leroy Quet, Denver, CO. Let $f(-1) = 1$, and for $m \geq 0$, let $f(m) = \prod_{k=0}^{(m-1)/4} (m - 4k)$. If $a_m = \frac{(-1)^{m+1}(8m^2 + 1)(f(2m - 3))^2}{2m(f(2m - 1))^2}$ for $m \geq 1$, show that

$$a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}} = \frac{2}{4 - \pi}.$$

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**SOLUTIONS**

**Comparison of Some Squared Distances**

10970 [2002, 854]. Proposed by Razvan Satnoianu, City University, London, U.K. Let $ABC$ be an acute triangle and let $P$ be a point in its interior. Denote by $a, b, c$ the lengths of the triangle’s sides, by $d_a, d_b, d_c$ the distances from $P$ to the triangle’s sides, and by $R_a, R_b, R_c$ the distances from $P$ to the vertices $A, B, C$ respectively. Show that

$$d_a^2 + d_b^2 + d_c^2 \geq R_a^2 \sin^2(A/2) + R_b^2 \sin^2(B/2) + R_c^2 \sin^2(C/2) \geq \frac{1}{3}(d_a + d_b + d_c)^2.$$

When is equality possible?

Solution by John G. Heuer, Grande Prairie, AB, Canada. Let $Q_a, Q_b, Q_c$ be the feet of the perpendiculars from $P$ to $BC, CA, AB$ respectively. The points $Q_b$ and $Q_c$ both lie on the circle with diameter $PA$, so the Extended Law of Sines and the Law of Cosines yield

$$R_a^2 = \frac{(Q_bQ_c)^2}{\sin^2 A} = \frac{d_a^2 + d_c^2 + 2d_bd_c \cos A}{\sin A}.$$

Since $d_b^2 + d_c^2 \geq 2d_bd_c$, with equality if and only if $d_b = d_c$, we have $R_a^2 \sin^2 A \leq (d_a^2 + d_c^2)(1 + \cos A)$; hence, $\frac{1}{2}(d_a^2 + d_c^2) \geq R_a^2 \sin^2(A/2)$. Summing this and the two analogous inequalities yields

$$d_a^2 + d_b^2 + d_c^2 \geq R_a^2 \sin^2(A/2) + R_b^2 \sin^2(B/2) + R_c^2 \sin^2(C/2),$$

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