Europ. J. Combinatorics (1994) **15**, 303–307

Frame Matroids and Biased Graphs

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A *frame matroid* is any submatroid of a matroid in which each point belongs to a line spanned by a fixed basis. A *biased graph* is a graph with certain polygons called *balanced*, no theta graph containing exactly two balanced polygons. We prove that certain matroids, called *bias matroids*, of biased graphs are identical to the finitary frame matroids. As an application we deduce two simple characterizations of frame matroids and some facts about planar forbidden minors for bias matroids.

A *biased graph* $\Omega = (V, E, \mathscr{B})$ is a graph (V, E) together with a *linear subclass* $\mathscr{B}$ of its polygons: a subclass such that, if in a theta subgraph two polygons are in $\mathscr{B}$, so is the third. Associated to a biased graph is a matroid $G(\Omega)$, its *bias matroid*, which is the natural common generalization of the polygon matroid $G(\Gamma)$ of a graph Γ (this being the case in which $\mathscr{B}$ contains all polygons), the bicircular matroid ($\mathscr{B}$ is empty) [4, 6], the even-cycle matroid [1], Dowling's geometries [2], and the matroids of networks with gains (see, for instance, [5]). Thus it is of considerable interest to know which matroids have the form $G(\Omega)$. One answer would be a characterization by forbidden minors (non-bias matroids the proper minors of which are bias matroids), since any minor of a bias matroid is a bias matroid. Here we take a step in that direction by proving that the bias matroids of a biased graph are identical to the finitary frame matroids, a *finitary frame matroid* being a submatroid of a finitary matroid M in which there is a basis V such that every point of M belongs to $\bigcup\{\bar{W}: W \subseteq V \text{ and } |W| \leq 2\}$, where $\bar{W}$ is closure in M. (That every bias matroid is a frame is trivial. The interesting point is that nothing can happen in a finitary frame but what is expressible by a biased graph.) From this result we draw some fairly strong conclusions about planar forbidden minors for bias matroids of biased graphs.

To define $G(\Omega)$: simplifying slightly (but not essentially), we call a subgraph or edge set of Ω *balanced* if each polygon in it belongs to $\mathscr{B}$. Letting $V(S)$ be the set of vertices of edges $e \in S$, where $S \subseteq E$, and $b(S)$ the number of connected components of $(V(S), S)$ that are balanced, we have for the rank function in $G(\Omega)$ the formula $r_{G(\Omega)}(S) = |V(S)| - b(S)$.

The simplification is the neglect of half and loose edges, which in matroid theory are equivalent to unbalanced and balanced loops. For this and other information about biased graphs, we refer the reader to [8]. For matroid theory, we refer to [7]. We reserve the term *circuit* for matroid circuits. The term *frame* is due to [3]. The hypothesis of finitarity could doubtless be weakened at some cost in complexity (cf. [8, Section II.5)].

THEOREM 1. *A finitary frame matroid is the bias matroid of a biased graph and conversely.*

PROOF. We do not make the usual requirement that V and E be disjoint. Instead, we permit overlap (essentially following [3, Section 7]). A vertex which is an edge should be thought of as an unbalanced loop (or equivalently a half edge) at itself. If $\Omega = (V, E, \mathscr{B})$ is a biased graph, we write $\Omega^{\bullet} = (V, E \cup V, \mathscr{B})$.

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The easy part is that $G(\Omega)$ is a finitary frame matroid. Look at $G(\Omega^{\bullet})$ with basis V, noting that it is finitary by [8, Theorem II.2.1].

Now, let M be a finitary frame matroid with respect to a basis $V \subseteq E(M)$, the point set of M. We may assume that M is simple. The first task is to find the right biased graph Ω. We naturally take $V(\Omega) = V$ and $E(\Omega) = E(M)$ (shortened to E), the endpoints of $e \in E \backslash V$ being the members of $V(e) = C_e \backslash e$, where C_e is the fundamental circuit of e with respect to V. For $\mathscr{B}$ we take the class of polygons C in the graph (V, E) that are dependent in M. Hence we have a biased graph if the theta-graph condition is satisfied. We need to prove that it is and that $G(\Omega) = M$. Note that any submatroid or contraction of a bias matroid is a bias matroid.

We say that $S \subseteq E$ is *M-balanced* if $\bar{S} \cap V = \emptyset$. Set $b_M(S)$ equal to the number of connected components of the graph $(V(S), S)$ that are M-balanced. It will turn out that M-balance is the same as balance, and that will suffice to prove the theorem. Observe that, for $W \subseteq V$, $\bar{W}$ is the set of edges induced by W.

LEMMA 1. *If S is a connected subset of E and $v \in V(S)$, then $\overline{S \cup v} = \overline{V(S)}$.*

PROOF. The proof is obvious. ∎

LEMMA 2. *For $S \subseteq E$, $r_M(S) = |V(S)| - b_M(S)$.*

PROOF. Let $(V(S), S)$ have M-balanced components $(B_1, S_1), \ldots, (B_k, S_k)$ and M-unbalanced ones $(B'_1, S'_1), \ldots, (B'_l, S'_l)$. By Lemma 1, $r(S_i) = |B_i| - 1$ and $r(S'_i) = |B'_i|$. Therefore $r(S) \leq \sum_i r(S_i) + \sum_i r(S'_i) = |V(S)| - k$.

On the other hand, let $v_i \in V(S_i)$ and $W = \{v_1, \ldots, v_k\}$. Then $\overline{S \cup W} = \overline{V(S)}$, so $r(S \cup W) = |V(S)|$. Hence $r(S) \geq |V(S)| - r(W)$. But $r(W) = k$. ∎

We regard a forest as an edge set (contained in $E \backslash V$), not as a subgraph.

LEMMA 3. *A forest of Ω is independent in M.*

PROOF. If F is a finite forest and W consists of one vertex from each tree of F, then $\overline{F \cup W} = \overline{V(F)}$ by Lemma 1. Consequently, $r(F) \geq |V(F)| - |W| = |F|$. ∎

LEMMA 4. *A polygon C in Ω is either independent, M-unbalanced and unbalanced, or a circuit, M-balanced and balanced.*

PROOF. The former case applies if $\bar{C} = \overline{V(C)}$. Otherwise, $\bar{C} \cap V(C) = \emptyset$ by Lemma 1. Then C is dependent because $r(C) < |V(C)|$. Any proper subset is a forest, and thus is independent. ∎

LEMMA 5. *An edge set is M-balanced iff it is balanced.*

PROOF. By Lemma 4, an M-balanced set is balanced.

Let S be balanced, but suppose that there is a vertex $v \in \bar{S} \cap V$. Then there is a circuit $D \subseteq S \cup v$ such that $v \in D$. Let $B = D \backslash v$. B cannot contain an unbalanced polygon because $B \subseteq S$, nor a balanced polygon because that would be a circuit (by Lemma 4). Thus B is a forest. Let $B_1, \ldots, B_k$ be its trees.

Since $v \in \bar{B}$, v lies in some $V(B_i)$, say, $V(B_1)$. It follows that the components of the graph $(V(D), D)$ have edge sets $B_1 \cup v, B_2, \ldots, B_k$. Of these, the first is certainly not M-balanced, so by Lemma 2 $r(D) = |V(D)| - b_M(D) > |V(D)| - k$. On the other hand, $r(B) = |B| = |V(B)| - k$, since B is a forest. But $V(D) = V(B)$ and $r(D) = r(B)$. This yields a contradiction. So no D can exist. ∎

LEMMA 6. *$\mathscr{B}$ is a linear subclass.*

PROOF. Consider a theta graph $C_1 \cup C_2$, where C_1 and C_2 are balanced polygons. Let C_3 be the third polygon in $C_1 \cup C_2$. Take $e_1 \in C_1 \backslash C_2$ and $e_2 \in C_2 \backslash C_1$. The set

$T = C_1 \cup C_2 \backslash \{e_1, e_2\}$ is a tree, and hence independent by Lemma 3, and $\bar{T} \supseteq C_1 \cup C_2$ because C_1 and C_2 are circuits by Lemma 4. But T is M-balanced, and hence so is $\bar{T}$, and then by Lemma 5 $\bar{T}$ is balanced. Consequently C_3 is balanced. ■

We now have everything that we need to prove the theorem. Ω is a biased graph, so M and $G(\Omega)$ are finitary matroids on the same set with the same rank function, $r(S) = |V(S)| - b(S)$ by Lemma 5. Therefore they are the same. □

Two simple characterizations of frame matroids are corollaries of the graphical representation. Here are the necessary definitions. A finitary *basic matroid* (M, V) is a matroid M together with a finitary matroid $M \cup V$, of which V is a basis. (V may overlap with $E(M)$.) For $S \subseteq E(M)$ we let

$$V(S) = S \cup \bigcup \{C_e \backslash e : e \in S \backslash V\},$$

which is the smallest subset of V the closure of which contains S. We call S *basically connected* if S is contained in one matroid component of $\overline{S \cup V}$. We let $b_{(M,V)}(S)$ be the number of matroid components T of $S \cup V(S)$ for which $\overline{S \cap T} \cap V = \varnothing$. Evidently, $b_{(M,V)} = b_M$ when (M, V) is a frame.

COROLLARY 1. *Given a finitary basic matroid (M, V), the following statements are equivalent*:
(*i*) *M is a frame matroid with respect to the basis V.*
(*ii*) *For every basically connected set $S \subseteq E(M)$, $\bar{S} \cap V$ is empty or spans S.*
(*iii*) *For every finite $S \subseteq E(M)$, $r_M(S) = |V(S)| - b_{(M,V)}(S)$.*

PROOF. That (i)$\Rightarrow$(ii), (iii) follows from Theorem 1, Lemma 5 (whence $b_M = b_\Omega$), and a few simple observations that we leave to the reader.

To prove the inverse implications we assume that M is not a frame with respect to V. Choose a point e for which $|V(e)| > 2$.

To disprove (ii) take $S = \{e\}$. Then $b_{(M,V)}(S) = 1$, so $|V(S)| - b_{(M,V)}(S) \geq 2 > r_M(S) = 1$, in contradiction to (ii).

To disprove (iii) take $S = \{e, v\}$, where $v \in V(e)$. S is obviously basically connected. If there were $w \in (\bar{S} \cap V) \backslash \{v\}$, then $\overline{\{v, w\}}$ would contain e, so $V(e) \subseteq \{v, w\}$, contrary to assumption. Therefore $\bar{S} \cap V = \{v\}$, contradicting (iii). □

Theorem 1 makes it easy in principle to find all the planar forbidden minors for bias matroids, because a planar matroid is a frame iff it has a covering by three non-concurrent lines (a *frame cover*). Although, unfortunately, there are too many planar forbidden minors to be readily enumerated, we can still give a fairly strong result.

Let Δ_6 denote a six-point matroid consisting of the vertices and midpoints of a triangle. Let Δ_7 be Δ_6 together with a point lying on the line connecting one vertex to the opposite midpoint. Let Π_{8a} consist of Δ_6 and two points added in planar general position, and let Π_{8b} consist of Δ_7 with one more point in planar general position. By $\Pi_{4,4,3}$ we mean any planar matroid which is the union of three concurrent lines of four, four and three points each and which does not contain F_7, the Fano plane. Π_{10a} is $G(K_4)$ with a point added in general position to each of the four three-point lines. Π_{10b} consists of a triangle of four-point lines l_1, l_2 and l_3, with a tenth point z and the non-vertex points $x_i, y_i \in l_i$ forming lines $x_i z y_{i+1}$, the subscripts taken modulo 3.

THEOREM 2. *The planar matroids which are forbidden minors for the class of bias matroids of biased graphs are the seven-point planes $U_{3,7}$ and F_7, the eight-point planes Π_{8a} and Π_{8b}, and other planar matroids of nine and ten points, such as several matroids of type $\Pi_{4,4,3}$ and Π_{10a} and Π_{10b}.*

PROOF. First we find the forbidden minors for the planar matroids which can be covered by two lines.

LEMMA 7. *A planar matroid can be covered by two lines iff it does not contain any of $U_{3,5}$, $G(K_4)$ and Δ_6.*

PROOF. A minimal two-line-uncoverable planar matroid N has at least five points. The complement of any line l must be a non-collinear set. Choose l to have maximum length and take a basis S of its complement. If $N \not\supseteq U_{3,5}$, the three lines determined by S must cover l or all but one point of l. Then $N \supseteq G(K_4)$ or Δ_6, respectively. ∎

A *long line* is a line of length four or more; a *trivial line* has only two points. In a matroid which has a three-line cover, every long line must belong to the cover.

LEMMA 8. *For a planar matroid M which can be covered by three lines to have no frame cover, it is necessary and sufficient that M contains F_7 or a $\Pi_{4,4,3}$.*

PROOF. Let M be covered by three concurrent lines. If one is trivial, M has a frame cover. If two are long and the third is non-trivial, the three-line covering is unique and M contains a $\Pi_{4,4,3}$ or F_7. If two are not long, but none is trivial, let l be the third line. The complement of l must be $U_{2,4}$. It follows that $M \supseteq F_7$. ∎

Now let M be a minimal planar, non-bias matroid, l a line in M, and S the complement of l. M obviously has at least seven points. If S has a two-line cover, by Lemma 8 M is F_7 or $\Pi_{4,4,3}$. Otherwise, by Lemma 7, S contains $S' = U_{3,5}$, $G(K_4)$ or Δ_6.

If M has a long line, let l be long. We can discard all but four of its points and all of $S \backslash S'$. Thus we have a non-bias submatroid of at most ten points, so $|E(M)| \leq 10$.

It is clear that all the examples mentioned in the theorem are minimal non-bias matroids. We need to show that the list is complete for seven and eight points. What prevents a planar M from having a frame cover is that, for any line l, its complement S contains $U_{3,5}$, $G(K_4)$ or Δ_6. So if $|E(M)| = 7$ and $M \neq U_{3,7}$, any non-trivial l belongs to a frame cover unless $M = F_7$. If $|E(M)| = 8$, the difficult case is where the longest line length in M is three and M is a minimal non-bias matroid. Let l be a three-point line. Then $S = U_{3,5}$. Since $M \not\supseteq U_{3,7}$, at least two points of l belong to non-trivial lines extending into S, say, l_1 and l_2. For M to be non-bias, l_1 and l_2 must meet. One can now easily show that M must be Π_{8a} or Π_{8b}. □

Not wanting to leave the impression that all forbidden minors for bias matroids are planar, we mention that the dual of F_7 is a forbidden minor. It is non-bias by [9, Proposition 3A], while it is well known that $F_7^{\perp}/\text{point} = G(K_4)$ and $F_7^{\perp}\backslash\text{point} = G(K_{3,3})$.

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Received 9 June 1992 and accepted 24 July 1993

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