LECTURES ON
GAIN GRAPHS AND HYPERPLANE ARRANGEMENTS

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Autumn, 2019 and Spring, 2020

These notes assume acquaintance with the elementary theories of hyperplane arrangements and matroids as in Richard P. Stanley's Park City lecture notes [9].

These notes are edited from transcriptions taken in my class Math 580A, "Hyperplane Arrangements and Matroids", by Michael Gottstein, Nicholas Lacasse, Andrew Lamoureux, Shuchen Mu, and James West, to whom I am grateful for their invaluable help.

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Lecture 0: Overview

1 November 2019
Notetaker: Andrew Lamoureux

Let K be a field, let $a \in K^\times$, and let h_{ij}^a be the hyperplane in K^n with equation $x_i = ax_j$. Also, define h_i to be the coordinate hyperplane $h_i = 0$. Note that $h_{ij}^a = h_{ji}^{a^{-1}}$. This property leads to the definition of a "gain graph".

Let Γ be a graph and $\mathfrak{G}$ a group. Orient the edges of Γ, and let $\vec{E}$ be the set of oriented edges. Denote by e^{-1} the edge e with its opposite orientation. A $\mathfrak{G}$-*gain graph* is $\Phi = (\Gamma, \varphi)$, where $\varphi : \vec{E} \to \mathfrak{G}$ satisfies $\varphi(e^{-1}) = \varphi(e)^{-1}$ for all oriented edges e.

The function φ is called a *gain function*, and $\varphi(e)$ is the *gain* of e.

Gain graphs are also allowed to have *half edges*, incident to a single endpoint. These are not the same as loops: a loop is an ordinary edge with two endpoints that happen to coincide. Half-edges are not assigned a gain.

An especially important and most studied example is *signed graphs*, for which the gain group has order 2; usually it is written multiplicatively ($\{\pm\}$ or $\{\pm 1\}$) but sometimes additively ($\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$), depending on the interest.

Note that we distinguish between "directed" edges and "oriented" edges. A directed edge has a single fixed direction, while an oriented edge has a preferred direction for notational purposes, that may be changed as is convenient. This is similar to orientation of, for example, a simplex in topology: one direction is preferred for notation, but the other may still be used.

Definition. Let $\Phi = (\Gamma, \varphi)$ be a $\mathfrak{G}$-gain graph, and let W be the walk $v_0 e_1 v_1 \cdots v_{l-1} e_l v_l$ in Γ. We define the *gain of* W to be $\varphi(W) := \varphi(e_1)\varphi(e_2)\cdots\varphi(e_l)$.

In particular, this defines the gain of a circle C. The value of $\varphi(C)$ depends, in general, on both the direction and (if $\mathfrak{G}$ is non-abelian) the initial vertex. However, the property that C has identity gain is independent of both these choices. A circle C whose gain is the identity of $\mathfrak{G}$ is called *balanced* or *neutral*; otherwise, C is *unbalanced* or *non-neutral*. We denote by $\mathscr{B}(\Phi)$ the set of balanced circles of Φ. Finally, Φ is *balanced* or *neutral* when Φ has no unbalanced circles or half edges (half edges are considered unbalanced).

(Though it may make linguistic sense to call an unbalanced gain graph "biased", we reserve that term for a separate notion of a *biased graph*, of which an example is $(\Gamma, \mathscr{B}(\Phi))$. Here, we are singling out a certain collection of circles.)

Two-term hyperplane arrangements. Suppose $\Phi = (\Gamma, \varphi)$ is a $K^\times$-gain graph with vertex set $V = \{v_1, \ldots, v_n\}$. From Φ we obtain a hyperplane arrangement $\mathscr{H}[\Phi]$ in K^n, called a "two-term arrangement", whose hyperplanes are $h_{ij}^{\varphi(e_{ij})}$ for each $e_{ij} \in \vec{E}$, an (oriented) edge between v_i and v_j, and h_i for each half edge e_i, incident to vertex v_i. I will discuss these hyperplane arrangements at length in Lectures 2–10.

Figure is an example of a gain graph Φ, with the gains beside the edges and the edges oriented for reading the gains:

Let $K = \mathbb{R}$, for example. As $\varphi(v_1 v_2) = 2$, the hyperplane h_{12}^2 with equation $x_2 = 2x_1$ is in $\mathscr{H}[\Phi]$. Similarly, $h_{23}^{1/5} = h_{32}^5$ and $h_2 \in \mathscr{H}[\Phi]$, where $h_{23}^{1/5}$ has equation $x_3 = \frac{1}{5}x_2$ and h_2 has equation $x_2 = 0$. The intersection lattice is in Figure 2.

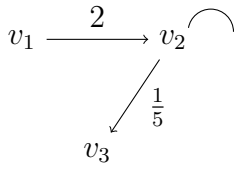

FIGURE 1. A gain graph.

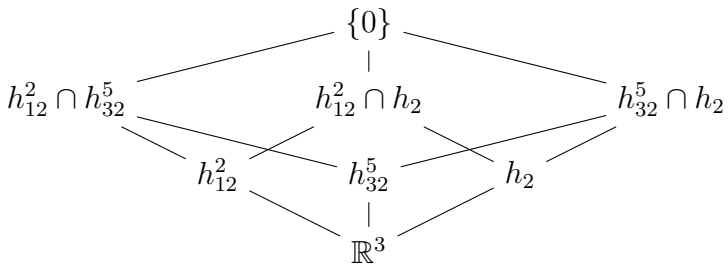

FIGURE 2. The intersection lattice of the arrangement associated to Figure .

Here

$$h_{12}^2 \cap h_{32}^5 = \{(x_1, 2x_1, \tfrac{2}{5}x_1) \mid x_1 \in \mathbb{R}\},$$
$$h_{12}^2 \cap h_2 = \{(0, 0, x_3) \mid x_3 \in \mathbb{R}\},$$
$$h_{32}^5 \cap h_2 = \{(x_1, 0, 0) \mid x_1 \in \mathbb{R}\}.$$

One may form a "completion" of Φ by adding implied hyperplanes. For example, given the equations $x_1 = 3x_2$, $x_2 = 2x_3$, and $x_3 = -x_1$, we can see that $x_1 = 6x_3 = -6x_1$, which implies $x_1 = 0$ as well as $x_2 = 0$ and $x_3 = 0$, so one may consider the arrangement to be completed by adding in these hyperplanes (as well as others implied by those equations, a large number if K is a large field). This would correspond to adding half edges incident to each vertex in the gain graph (and edges e_{ij} with all possible gains). I will discuss this notion of completion later in terms of closure within a given gain graph.

Affinographic hyperplane arrangements. Suppose now that the gain group is K^+. We associate to an edge e between v_i and v_j the hyperplane $a_{ij}^{\varphi(e_{ij})} : x_j - x_i = \varphi_{ij}$; these hyperplanes forms an arrangement $\mathscr{A}[\Phi]$ that we call "affinographic" in the affine space $\mathbb{A}^n(K)$. Its combinatorics is closely related to that of a two-term arrangement. I will discuss affinographic arrangements in Lectures 11 and 12.

Matroids. The usual matroid on a graph $\Gamma = (V, E)$ has ground set E, independent sets the edge sets of forests, and circuits the edge sets of circles in Γ. This is generally known nowadays as the *graphic matroid* of Γ, though it has other names; e.g., Stanley calls it the "bond matroid" (and—just to add to the confusion—it also has been called the "cycle matroid").

Notation. We consistently employ the following notations: K is a field, $\Gamma = (V, E)$ is a graph, $\Phi = (\Gamma, \varphi)$ is a gain graph and $\Omega = (\Gamma, \mathscr{B})$ is a biased graph with underlying graph

Γ, n is the dimension of a hyperplane arrangement and the order of the graph Γ, whence also of Φ and Ω.

Lecture 1: Gains

4 November 2019
Notetaker: Nicholas Lacasse

We begin with some preliminary notation and definitions. We use $\mathfrak{G}$ to denote a group and ε to denote its identity element. We will typically use Γ for a graph. The reader may be familiar with graphs that have loops and multiple edges (each of which have two ends, each incident with one vertex), which we call *ordinary edges*; but it is less likely that the reader has encountered half edges or loose edges. A *half-edge* is an edge with one end, which is incident to one vertex. A loose edge is an edge with no ends and no incident vertices.

For a graph Γ we denote its edge set by E and its vertex set by V.

Definition 1. A *gain graph*, Φ, is a pair (Γ, φ) where Γ is a graph and $\varphi : \vec{E} \to \mathfrak{G}$ is a function from the oriented ordinary edges of Γ to a group $\mathfrak{G}$ that satisfies $\varphi(e^{-1}) = \varphi(e)^{-1}$, where e^{-1} denotes e in the opposite orientation. We call φ a *gain function*.

This should be explained more fully. In a gain graph, each edge (with the exception of loose edges) has two possible orientations. Below we show the two possible orientations on links, loops, and half edges and we show the unoriented loose edge.

These orientations matter in order to define the value of the gain of an edge, which is inverted by reversing orientation. We need an arbitrary orientation so that we may define φ. In light of this, we define the set $\vec{E}$ to be the set of (arbitrarily) *oriented* edges of Γ. Thus when we write e^{-1} we mean the opposite direction of the arbitrarily chosen orientation. The gain function can be regarded as defined on $\vec{E}$ and therefore on E with the orientations left implicit, since knowing the gain on either orientation of an edge determines it on the opposite orientation.

Note that the edge orientations are not fixed, so we do not have a directed graph, in which edges have fixed directions.

Consider a gain graph $\Phi = (\Gamma, \varphi)$ and let $W = v_0 e_1 v_1 e_2 \cdots e_l v_l$ be a walk in Γ. We define the *gain of* W to be $\varphi(W) := \varphi(e_1)\varphi(e_2)\cdots\varphi(e_l)$. A *circle* is a 2-regular connected graph (or its edge set). A circle C has a gain $\varphi(C)$ which may depend on the initial point and direction; e.g., if $C = e_1 e_2 \cdots e_k$ then we could also write C as $e_3^{-1} e_2^{-1} e_1^{-1} e_k^{-1} e_{k-1}^{-1} \cdots e_4^{-1}$, and there is no guarantee that $\varphi(e_1 e_2 \cdots e_k) = \varphi(e_3^{-1} e_2^{-1} e_1^{-1} e_k^{-1} e_{k-1}^{-1} \cdots e_4^{-1})$. However, if $\varphi(C) = \varepsilon$, then $\varphi(C) = \varepsilon$ for any initial point and direction.

Proposition 2. *Let C be a circle in a gain graph Φ. If $\varphi(C) = \varepsilon$, then $\varphi(C)$ is independent of which vertex we start at and in which direction we traverse the circle.*

We are most interested in which circles have $\varphi(C) = \varepsilon$ and which do not. Thus, by proposition 2, we are justified in writing $\varphi(C)$.

Definition 3. Let $\mathscr{B}(\Phi) := \{\text{circles with gain } \varepsilon\}$. For $C \in \mathscr{B}(\Phi)$, we say C is *balanced* or *neutral*.

Definition 4. We call Φ *balanced* (or *neutral*) if every circle of Φ is balanced and Φ has no half edges.

We also call a subgraph of Φ, or an edge set, balanced if every circle in it has gain ε and it has no half edges.

Definition 5. A theta graph is the union of three internally disjoint paths with the same two end points.

Definition 6. We denote the free group on a set E by $\mathfrak{F}(E)$.

A gain function φ defines a homomorphism $\varphi : \mathfrak{F}(\vec{E}) \to \mathfrak{G}$ in the "obvious" way. That is, $\varphi(e_1^{\pm 1} e_2^{\pm 1} \cdots e_l^{\pm 1}) = \varphi(e_1)^{\pm 1}\varphi(e_2)^{\pm 1} \cdots \varphi(e_l)^{\pm 1}$. In view of the remarks after Definition 1, we can simply write $\mathfrak{F}(E)$ for this free group.

Proposition 7. *$\mathscr{B}(\Phi)$ has the property that no theta subgraph contains exactly two balanced circles.*

Proof. Let P_1, P_2, P_3 be three internally disjoint paths from v to w. This subgraph has three circles, $P_1P_2^{-1}, P_2P_3^{-1}$, and $P_1P_3^{-1}$. Since φ defines a homomorphism on $\mathfrak{F}(E)$ to $\mathfrak{G}$ and $P_1P_3^{-1} = (P_1P_2^{-1})(P_2P_3^{-1})$, then $\varphi(P_1P_3^{-1}) = \varphi(P_1P_2^{-1})\varphi(P_2P_3^{-1})$. Suppose without loss of generality that circles $P_1P_2^{-1}$ and $P_2P_3^{-1}$ are balanced, i.e., $\varphi(P_1P_2^{-1}) = \varphi(P_2P_3^{-1}) = \varepsilon$. Then $\varphi(P_1P_3^{-1}) = \varphi(P_1P_2^{-1})\varphi(P_2P_3^{-1}) = \varepsilon\varepsilon = \varepsilon$. □

Definition 8. Let $\mathscr{C}(\Gamma)$ be the class of all circles in Φ. Let $\mathscr{B} \subseteq \mathscr{C}(\Gamma)$. We call $\mathscr{B}$ a *linear class* if no theta subgraph of Γ contains exactly two circles in $\mathscr{B}$. (The notion of a linear class originated in Tutte's theory of matroids.)

Definition 9. Let Γ be a graph and let $\mathscr{B}$ be a linear class of circles. We call $(\Gamma, \mathscr{B})$ a *biased graph*. We call a circle *balanced* if it belongs to $\mathscr{B}$.

All the matroid theory of gain graphs will generalize to biased graphs. However, some proofs are more complicated for biased graphs, because gains provide helpful extra structure beyond the balanced circle class. A good example of this is switching.

Definition 10. *Switching* $\Phi = (\Gamma, \varphi)$ means taking a function $\zeta : V \to \mathfrak{G}$ and replacing φ by φ^ζ, which is defined by $\varphi^\zeta(e_{uv}) = \zeta(u)^{-1}\varphi(e_{uv})\zeta(v)$. By Φ^ζ we mean the gain graph (Γ, φ^ζ) obtained by switching Φ by ζ.

We say Φ and Φ^ζ are *switching equivalent*, written $\Phi \sim \Phi^\zeta$. This is clearly an equivalence relation (Exercise!). The equivalence class of Φ is called its *switching equivalence class* or, more simply, *switching class*, and is denoted by $[\Phi]$.

Proposition 11. *For a walk W from u to v, $\varphi^\zeta(W) = \zeta(u)^{-1}\varphi(W)\zeta(v)$. For a closed walk W, $\varphi^\zeta(W)$ is the conjugate of $\varphi(W)$ by $\zeta(u)$.*

Proposition 12. *A circle is balanced in Φ^ζ if and only if it is balanced in Φ. In other words,*

$$\mathscr{B}(\Phi^\zeta) = \mathscr{B}(\Phi).$$

Here is one use for switching:

Proposition 13. *Let $\Phi = (\Gamma, \varphi)$. Let F be a forest in Γ. Let $\tau : F \to \mathfrak{G}$ be any function. Then we can switch φ so that $\varphi^\zeta|_F = \tau$.*

Proof. Exercise! □

In particular, we often want to pick a maximal forest F of Φ and switch φ so that it is the identity on F.

Lecture 2: Independence via the Two-Term Hyperplane Arrangement

6 November 2019
Notetaker: Shuchen Mu

Definition 14. Take a gain graph $\Phi = (\Gamma, \varphi)$ for which φ takes values in $K^\times$, the multiplicative group of the field K. The hyperplane arrangement associated to the graph, $\mathscr{H}[\Phi]$, is in K^n, where $n =$ number of vertices in the graph. It is $\mathscr{H}[\Phi] = \{h(e) : e \in E\}$, where

$$\begin{aligned} h(e_{ij}) : x_j = x_i\varphi(e_{ij}) \qquad & \text{if } e_{ij} \text{ is the edge from } v_i \text{ to } v_j, \\ h(e_i) : x_i = 0 \qquad & \text{if } e_i \text{ is a half edge with vertex } v_i, \\ h(e) : 0 = 0 \qquad & \text{if } e \text{ is a loose edge.} \end{aligned}$$

Because the hyperplane equations all have the form $x_j = x_i a$, with two terms, I call this a "two-term arrangement".

For any edge set S, we define $h(S) := \{h(e) : e \in S\}$. Thus, $\bigcap h(S) := \bigcap_{e\in S} h(e)$.

This definition extends to division rings. I will focus on fields so as to keep things simple. In the noncommutative extension the gains must be written on the right; I will follow that convention throughout so the extension will not require rewriting.

The arrangement $\mathscr{H}[\Phi]$ of a gain graph is homogeneous. Later—in Lecture 11—we will meet another kind of arrangement, associated to a gain graph in a different way, which is affine (and for which I reserve the letter $\mathscr{A}$).

Let's consider the gain of a circle. We defined the gain of a walk, say $W = e_{01}e_{12}\cdots e_{(l-1)l}$, as

$$\varphi(W) := \varphi(e_{01})\varphi(e_{12})\cdots\varphi(e_{(l-1)l}).$$

In a circle C, we choose one of the vertices as v_0 and start labelling all its vertices in one direction as $v_0, v_1, \ldots, v_l = v_0$. This defines the gain of C, since we can write $C = e_{01}e_{12}\cdots e_{(l-1)l}$, where v_0 is just another name for v_l. (Recall that the coordinate x_i of $x \in K^n$ is associated with the vertex v_i; we have assumed this labelling of C just to keep the notation simple. Since v_0 is the same as v_l, x_0 is merely another notation for x_l.) The hyperplanes associated to the edges on the circle are:

$$\begin{aligned} h(e_{01}) : x_1 &= x_0\varphi(e_{01}), \\ h(e_{12}) : x_2 &= x_1\varphi(e_{12}), \\ &\cdots \\ h(e_{(l-1)l}) : x_l &= x_{l-1}\varphi(e_{(l-1)0}). \end{aligned}$$

(For technical reasons we admit the whole space K^n as the "degenerate hyperplane".) The sequence of equations implies $x_0 = x_0\varphi(C)$.

Case 1. C is balanced, i.e., $\varphi(C) = 1$. If a point

$$x = (x_1, x_2, \ldots, x_n) \in \bigcap h(C \setminus \{e_{l-1,l}\}),$$

(recall that x_0 is another name for x_l), then $x_{l-1} = x_0\varphi(e_{01})\varphi(e_{12})\cdots\varphi(e_{(l-2)(l-1)})$, so $x_0 = x_{l-1}\varphi(e_{(l-1)l})$, so $x \in h(e_{(l-1)l})$. Therefore, $\bigcap_{i=1}^{l-1} h(e_{i-1,i}) \subseteq h(e_{l-1,l})$, so $h(C)$ is dependent.

Case 2. If C is unbalanced, then $x_0 = x_0\varphi(C)$ where $\varphi(C) \neq 1$, so $x_0 = 0$. It follows that $x_1 = 0, x_2 = 0, \ldots$, so

$$\bigcap h(C) = \{x \in K^n \mid x_i = 0 \ \forall \ v_i \in V(C)\}.$$

Between Cases 1 and 2 we have proved

Lemma 15 (Dependence of a Circle)**.** *If C is a balanced circle, then $h(C)$ is a dependent set of hyperplanes, but $h(C \setminus \{e\})$ is an independent set for every edge $e \in C$.*

If C is an unbalanced circle, then $h(C)$ is an independent set of hyperplanes.

Now we turn our attention to forests.

Lemma 16. *Let F be a forest in Φ. Then $h(F)$ is an independent set of hyperplanes.*

Proof. Here we think of a forest as an edge set.

Case 1. If there is only one edge in the forest, then it forms an independent singleton because its hyperplane is independent.

Case 2. Suppose all forests with a number of edges $\leq k$ have independent image under h. Let F be a forest that consists of $k+1$ edges.

Take a pendant edge e_{kl}, so there is no other edge than e_{kl} in the forest incident to v_l. Therefore the only defining equation of edges in the forest that involves x_l is $x_l = x_k\varphi(e_{kl})$. Let $F' = F \setminus \{e_{kl}\}$. Denote the defining vector of $h(e)$ by u_e. By the preceding discussion,

$$u_{e_{kl}} \notin \operatorname{span}\{u_e \mid e \in F'\},$$

so

$$\operatorname{rk}\operatorname{span}\{u_e \mid e \in F'\} < \operatorname{rk}\operatorname{span}\{u_e \mid e \in F\}.$$

By assumption, $h(F')$ is independent, so $\operatorname{rk}\operatorname{span}\{u_e \mid e \in F'\} = \#F'$. Hence

$$\operatorname{rk}\operatorname{span}\{u_e \mid e \in E(F)\} \geq \#F = \#F' + 1.$$

Thus, $h(F)$ is independent. □

Next, we need a little matroid theory.

Definition 17. The direct sum of two matroids, $M_1 \oplus M_2$, is defined by

$$E(M_1 \oplus M_2) = E(M_1) \uplus E(M_2),$$
$$\operatorname{rk}_{M_1 \oplus M_2}(X) = \operatorname{rk}_{M_1}(X \cap E(M_1)) + \operatorname{rk}_{M_2}(X \cap E(M_2)).$$

Let's look at independence. Recalling that $\operatorname{rk}(X) \leq \#X$, we cna see that

$$\operatorname{rk}_{M_1}(X \cap E(M_1)) + \operatorname{rk}_{M_2}(X \cap E(M_2)) = \#X$$

if and only if

$$\operatorname{rk}_{M_1}(X \cap E(M_1)) = \#[X \cap E(M_1)],$$

and also

$$\operatorname{rk}_{M_1}(X \cap E(M_1)) = \#[X \cap E(M_1)].$$

That is, X is independent in $M_1 \oplus M_2$ if and only if its M_1 part and its M_2 part are independent in their respective matroids. This is a characteristic of the direct sum; if it is true for every X, then we have the direct sum of M_1 and M_2.

Consider how this applies to a hyperplane arrangement.

Lemma 18. *Let $\mathscr{B}, \mathscr{C}$ be arrangements in K^n, such that the coordinates of the hyperplanes in $\mathscr{B}$ are all different from those of the hyperplanes in $\mathscr{C}$. Then $\mathscr{M}(\mathscr{B} \cup \mathscr{C}) = \mathscr{M}(\mathscr{B}) \oplus \mathscr{M}(\mathscr{C})$.*

Proof. Since the coordinates of hyperplanes in $\mathscr{B}$ are different from those in $\mathscr{C}$, $\mathscr{B} \cup \mathscr{C} = \mathscr{B} \cup\!\!\cdot\, \mathscr{C}$.

Let X be an independent set in the matroid $\mathscr{M}(\mathscr{B} \cup \mathscr{C})$. There are no linear relations between defining vectors in the set $\mathscr{B}$ and those in the set $\mathscr{C}$. Thus, if $X \cap \mathscr{B}$ is independent in $\mathscr{M}(\mathscr{B})$ and $X \cap \mathscr{C}$ is independent in $\mathscr{M}(\mathscr{C})$, then X is independent in $\mathscr{M}(\mathscr{B}) \oplus \mathscr{M}(\mathscr{C})$, and conversely. Thus, $\mathscr{M}(\mathscr{B} \cup \mathscr{C}) = \mathscr{M}(\mathscr{B}) \oplus \mathscr{M}(\mathscr{C})$. □

Lemma 19. *Suppose $D \subseteq E(\Phi)$ is such that each component of D is a tree or contains only one circle, which is unbalanced, or a half edge. Then $h(D)$ is independent in $\mathscr{M}(\mathscr{H}[\Phi])$.*

Proof. let K be a component that contains one unbalanced circle C. Then by Lemma 15, $\bigcap_{e \in E(C)} h(e) = \{x : x_i = 0 \ \forall \ v_i \in V(C)\}$. Since K is connected, each vertex $v_j \in V(K)$ is linked to C by a path, say P from $v_i \in V(C)$ to v_j. So if $x \in \bigcap h(E(K))$, then $x_j = x_i \varphi(P) = 0$. Therefore, $x_j = 0$ for every $v_j \in V(K)$. On the other hand, if $x_j = 0$ for all $v_j \in V(K)$, then certainly $x_i \in \bigcap h(E(K))$. This shows that $\bigcap h(E(K)) = \{x : x_j = 0 \ \forall \ v_j \in V(K)\}$, which is of codimension $\#V(K)$, which $= \#E(K)$, so $E(K)$ is independent.

Let K be a component with a half edge attached to a vertex v_i, so $x_i = 0$. Since K is connected, by similar reasoning we conclude that $E(K)$ is independent.

By Lemma 16, if K is a tree, then $h(E(K))$ is independent.

As we saw in Lemma 18, this implies that $h(D)$ is independent. □

Lemma 19 begins to tell us how to define independent sets in the frame matroid of a $K^\times$-gain graph. In particular, if an edge set contains a balanced circle, then it is dependent.

Direct sums of arrangements. In connection with direct sums of matroids we should mention direct sums of hyperplane arrangements. If we have an arrangement $\mathscr{B}'$ in K^m and another one, $\mathscr{C}'$, in K^n, then there is a *direct sum* $\mathscr{B}' \oplus \mathscr{C}'$ in K^{m+n} whose hyperplanes are $g \oplus K^n$ for each $g \in \mathscr{B}'$ and $K^m \oplus h$ for each $h \in \mathscr{C}'$. It is easy to see that the flats of $\mathscr{B} \oplus \mathscr{C}$ are the combinations of flats of $\mathscr{B}'$ and $\mathscr{C}'$; in precise form, the sets

$$s \oplus t = \{(x, y) \in K^m \times K^n \mid x \in s, \ y \in t\}$$

for each pair $(s, t) \in \mathscr{L}(\mathscr{B}') \times \mathscr{L}(\mathscr{C}')$.

We connect this to the situation of Lemma 18 by defining $\mathscr{B} := \{g \oplus K^n \mid g \in \mathscr{B}'\}$ and $\mathscr{C} := \{K^m \oplus h \mid h \in \mathscr{C}'\}$. Then $\mathscr{B}$ and $\mathscr{C}$ are the $\mathscr{B}$ and $\mathscr{C}$ of Lemma 18. Conversely, if in the lemma, where we are in K^n, the coordinates of hyperplanes in $\mathscr{B}$ are among the first m coordinates of K^n and those of the hyperplanes in $\mathscr{C}$ are among the last $n - m$ coordinates of K^n, then let $\mathscr{B}'$ consist of the hyperplanes in K^m with the same equations as those in $\mathscr{B}$ and let $\mathscr{C}'$ consist of the hyperplanes in K^{n-m} with the same equations as those in $\mathscr{C}$. Then the arrangement $\mathscr{B} \cup \mathscr{C} = \mathscr{B}' \oplus \mathscr{C}'$. Thus, by Lemma 18 (but also obviously), we have $\mathscr{M}(\mathscr{B}' \oplus \mathscr{C}') = \mathscr{M}(\mathscr{B}) \oplus \mathscr{M}(\mathscr{C})$.

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Lecture 3: Independence, concluded

8 November 2019
Notetaker: Michael Gottstein

We finish the proof of Theorem 3. First, a few new definitions.

Definition 20. A *unicyclic graph* is a connected graph with exactly one circle. A 1-*tree* is a tree with one extra edge, either a half edge or forming a unicyclic graph. A *pseudotree* is a tree or a 1-tree. A *pseudoforest* is a graph whose components are pseudotrees. (A 1-*forest*, similarly, is a graph whose components are 1-trees, but we have not much use for this.)

Definition 21. A gain graph is *contrabalanced* if it has no loose edges and every circle is unbalanced; i.e., it has no loose edges or balanced circles.

A forest, and only a forest, is both balanced and contrabalanced.

Theorem 22. *Let Φ be a $K^\times$-gain graph, Let M be the matroid on $E(\Phi)$ that corresponds to $\mathscr{M}(\mathscr{H}[\Phi])$. The independent sets in M are (the edge sets of) the contrabalanced pseudoforests in Φ.*

For the proof, recall that Lemma 3 says an edge set each of whose components is a tree, an unbalanced unicycle, or a tree with a single half edge is independent in the matroid M of the gain graph Φ that corresponds to the hyperplane arrangement $\mathscr{H}[\Phi]$ over a field K. I.e., any contrabalanced pseudoforest is independent in M.

So we must prove every independent edge set is a contrabalanced pseudoforest. We'll prove the contrapositive:

Lemma 23. *The dependent sets are the edge sets that have a single component with at least 2 circles, at least 2 half edges, or at least one circle and one half edge.*

Proof. Since a set containing a dependent set is dependent we only need to prove that a connected edge set that contains, two circles, two half edges, or a circle and half edge is dependent in M.

Suppose S is such an edge set. If S contains a balanced circle then we know it is dependent by Case 1 of the previous treatment of a circle. So we may assume every circle in S is unbalanced.

Case 1. S contains an unbalanced circle or half edge, C_1, and another one, C_2, that share at most one vertex. There must be a minimal path P connecting C_1 and C_2 (see Figure 3):

INSERT PHOTO

Figure 3. An edge set that contains two unbalanced circles or half edges.

By Case 2 of circles (Lecture 2) or by definition of a half-edge hyperplane $h(e)$, C_1 forces $x_i = 0$ and C_2 forces $x_j = 0$.

Indeed, write $P = w_0e_1w_1....w_{l-1}e_lw_l$ so that in $\mathscr{H}[\Phi]$,

$$x_{w_l} = x_{w_{l-1}}\varphi_{l-1,l}(e_l) = ... = x_{w_0}\varphi(P).$$

But $x_{w_0} = 0$ from C_1, therefore $x_{w_l} = 0$.

We will show that some $e \in S$ has the property that

$$h(e) \supseteq \bigcap_{f \in S \setminus e} h(f).$$

(Every $e \in S$ has that property, but we don't need that.)

Pick e to be an edge in C_2 incident with v_j so $e = e_{jk}$ or the half edge e_j. With the equations $x_k = x_j \varphi_{jk}(e)$ for a circle and $x_j = 0$ a half edge.

Then we consider the equations separately.

In the half edge case $x_j = 0$ (consider C_1 and P) in $\bigcap h(C_1 \cup P) = \bigcap h(S \setminus e_j)$, therefore $h(S)$ is a dependent set of hyper planes.

In the circle case $x_j = 0$ and $x_k = 0$ (consider C_1 and the path $P \cup (C_2 \setminus e)$), then $x_k = x_j \varphi_{jk}(e)$ is satisfied (with $x_k = x_j = 0$) by $\bigcap h(S \setminus e)$; therefore $h(S)$ is a dependent set of hyperplanes.

This solves the problem when S contains two unbalanced circles/half edges with at most one vertex in common.

Case 2: Now we will consider the case where S contains two circles with at least two common vertices.

We treat first the case where S is a theta graph. We have 3 unbalanced circles, say $C_{12} = P_1 P_2^{-1}$, $C_{23} = P_1 P_3^{-1}$, $C_{13} = P_1 P_3^{-1}$ given by the internally disjoint paths of the theta graph, P_1, P_2, P_3, which all start and end at v_i and v_j, respectively. Then C_{12}, being unbalanced, implies $x_i = x_j = 0$. Let $e = e_{jk}$ be the edge in p_3 at v_j Then $C_{12} \cup (P_3 \setminus e)$ implies $x_k = 0$ thus $x_k = 0$ in $\bigcap h(C_{12} \cup (P_3 \setminus e)) = \bigcap h(S \setminus e)$. Similarly, $x_j = 0$. Therefore $h(e) \supseteq h(S \setminus e)$, so the theta graph is dependent in M. So if S contains a theta graph, it is dependent.

Note that we have been assuming S is connected. Let $\xi(S)$ be the *cyclomatic number* of S, defined as the smallest number of edges that when deleted leave a tree spanning $V(S)$, i.e., the minimum number of edges whose deletion leave a forest (therefore a tree). We have $\xi(S) > 1$ because C_1 and C_2 exist. If $\xi(S) = 2$ then S contains the theta graph $C_1 \cup C_2$ (because those two circles have at least two common vertices), so we are done. If $\xi(S) > 2$ then we can delete $\xi(S) - 2$ edges to get a connected subgraph S' with $\xi(S') = 2$, which contains a theta graph or handcuff (by easy graph theory) and is therefore dependent.

So we have finished the proof modulo some graph-theoretic detail. $\square$

Lecture 4: The Frame Matroid

11 November 2019
Notetaker: Jimmy West

In the last lecture, we analyzed the matroid $\mathscr{M}(\mathscr{H}[\Phi])$ in terms of the independent sets of Φ. Theorem 22 showed that the independent sets are the same as the contrabalanced pseudoforests.

Definition 24. Let $\mathbf{F}(\Phi)$ be the matroid on a $K^\times$-gain graph Φ, with independent sets given by Theorem 22. This is called the *frame matroid* of Φ.

The reader will easily see that the ordinary graphic matroid of a graph Γ is the frame matroid of $\varepsilon\Gamma$, by which I mean the gain graph whose gains are all the group identity. Thus, the frame matroid generalizes the graphic matroid. In fact, the frame matroid of any balanced gain graph (or biased graph) is the graphic matroid of the underlying graph.

At present, we know this is a matroid only because it is the matroid of a hyperplane arrangement. We will not prove that the same definition gives a matroid on any gain graph and, most generally, any biased graph; that is true but too lengthy for here.

Theorem 25. *For the $K^\times$-gain graph Φ, let $S \subseteq E(\Phi)$. Then,*

(1) $\mathrm{rk}(S) = n - b(S)$.
(2) The circuits of $F(\Phi)$ are all the edge subsets that belong to any of the following three categories:
  (a) Balanced circles and loose edges.
  (b) Contrabalanced handcuffs.
  (c) Contrabalanced theta graphs.
(3) The closure of S is given by

$$\mathrm{cl}(S) = (E{:}V_0(S)) \cup \mathrm{bcl}(S{:}V_0(S)^c).$$

(4) S is closed if S is equal to its closure; i.e., the union of unbalanced components is an induced subgraph of Φ and each balanced component is balance-closed.

To explain the notation we give some new definitions.

Definition 26. For $S \subseteq E$, we define $b(S)$ to be the number of balanced components of the spanning subgraph (V, S). Isolated vertices are included as balanced components because they are balanced. However, loose edges are not included as they are not considered components because they have no vertices; we might say they are not "vertex components", although they are "edge components".

We define $V_0(S)$ to be the set of vertices that are contained in unbalanced components of S.

Definition 27. For $X \subseteq V$ and $S \subseteq E$, the *induced subset of S on X*, denoted by $S{:}X$, is defined as the set $\{e \in S :$ all endpoints of e are in X, and e has an endpoint in $X\}$. Since we want every edge in $S{:}X$ to have an endpoint in X, loose edges are not included in an induced subset.

Definition 28. The *balance-closure of* S, denoted by $\operatorname{bcl}(S)$, is given by

$$S \cup \{e \in E : \exists \text{ balanced circle } C \text{ such that } e \in C \subseteq S \cup \{e\}\} \\ \cup \{\text{loose edges}\},$$

which is equal to

$$S \cup \{e \in E : \exists \text{ path } P \text{ in } S \text{ joining the endpoints of } e \\ \text{such that } P \cup \{e\} \text{ is a balanced circle}\} \\ \cup \{\text{balanced loops}\} \cup \{\text{loose edges}\}.$$

In both sets the loose edges could be omitted if we consider loose edges to be balanced circles.

An edge set S is *balance-closed* if $S = \operatorname{bcl}(S)$.

Before we begin the proof of the theorem, there are two notes about the balance-closure of S. First, we have not said that balance-closure satisfies all three conditions to be an abstract closure. Clearly, $S \subseteq \operatorname{bcl}(S)$ and balance-closure is weakly increasing (i.e., adding edges to S cannot remove an edge from $\operatorname{bcl}(S)$). However, we do not know that $\operatorname{bcl}(\operatorname{bcl}(S)) = \operatorname{bcl}(S)$. In fact, it may not be; there is a biased-graph example in [13, Part I, Proposition 3.5]. We show that this example is the bias of a gain graph with any nontrivial gain group.

Example 29. The underlying graph has $V = \{v, w, u_1, u_2\}$ and $E = \{e_1, e_2, f_1, f_2, g_1, g_2, h\}$; the endpoints of the edges are e_i:vu_i, f_i:vw, g_i:u_iw, h:u_1u_2. Assign gain ε to all edges except that $\varphi_{vu_2}(e_2) = \varphi_{vw}(f_2) = \alpha$ where $\alpha \neq \varepsilon$. Then the balanced circles are $e_if_ig_i$, $e_if_ig_jh$, g_1g_2h where $j \neq i$. (The verification is left to the reader.)

Now let $S = \{e_1, e_2, f_1, f_2\}$. Then $\operatorname{bcl} S = S \cup \{g_1, g_2\} = E \setminus \{h\}$ and $\operatorname{bcl}(\operatorname{bcl} S) = E$.

Secondly, the balance-closure of a balanced set must also be balanced, because it contains no unbalanced circles or half edges. This requires proof. This is a case where the proof for gain graphs is very simple while that for biased graphs uses advanced graph theory (according to [13, Part I, Proposition 3.1]).

Lemma 30. *If S is balanced, then* $\operatorname{bcl}(S)$ *is balanced.*

Proof. Switch so S has all identity gains. Then an edge added to S by balance-closure must also have identity gain. It follows that $\operatorname{bcl}(S)$ is balanced. □

We begin the proof of Theorem 25 with part (2).

Proof of Part (2). Let D belong to one of the categories described in this part and let $e \in D$. From Theorem 1, we know that D is dependent and that $D \setminus e$ is independent. Independence comes from the fact that if we were to remove any edge from D, this would result in a contrabalanced pseudoforest. Therefore D is a circuit.

Conversely, let D be a circuit. Then D is dependent. By Theorem 1, D must contain a balanced circle or two unbalanced circles/half edges that are in the same component of D (since if they were in different components, they would not make D dependent). In Theorem 1 we already showed that this subset D' of D is dependent and is either a balanced circle or a contrabalanced handcuff or theta graph. By minimality, $D = D'$. □

Proof of Part (1). From the definition of $\mathbf{F}(\Phi)$ (as corresponding to $\mathscr{M}(\mathscr{H}[\varphi])$) we have $\operatorname{rk}(S) = \operatorname{rk}(\bigcap h(S))$, as the right-hand side is the rank function in the hyperplane arrangement. This is then equal to $\operatorname{codim} \bigcap h(S)$. We know that $\operatorname{rk}(S) = \operatorname{rk}(I)$ for any maximal

contrabalanced pseudoforest I because it is a maximal independent set. Thus we also have $\operatorname{rk}(S) = \operatorname{rk}(I) = \operatorname{codim} \bigcap h(I)$.

The following is a diagram of the structure of I.

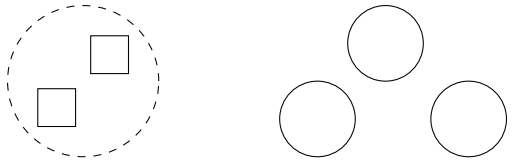


FIGURE 4. Structure of I. Each box represents an unbalanced component R_i of I and each circle represents a balanced component I_i of I. The dashed circle represents $V_0(S)$.

We can break up I into its unbalanced components and its balanced ones. Collectively, the unbalanced components are $I{:}V_0(S)$. For the balanced components, each component must be a tree because I is independent. (We are able to ignore loose edges as they correspond to the degenerate hyperplane.)

For an unbalanced component R_i, we found in the study of circles and the proof of Theorem 1 that $h(R_i) \implies x_j = 0$ for all $v_j \in V(R_i)$. Hence, we have a contrabalanced 1-tree. Therefore $\operatorname{codim} \bigcap h(R_i)$ will be equal to the number of coordinates of the hyperplane equations in $h(R_i)$. However, this is just the number of vertices of R_i.Therefore $\operatorname{rk}(R_i) = |V(R_i)|$.

Next, we consider a balanced component I_i. It is a tree, by Theorem 1. All coordinates of $\bigcap h(I_i)$ are determined through the equations of the tree by fixing any one coordinate arbitrarily (see Figure 5). Therefore $\operatorname{codim} \bigcap h(I_i) \geq |V(I_i)| - 1 = |E(I_i)$.

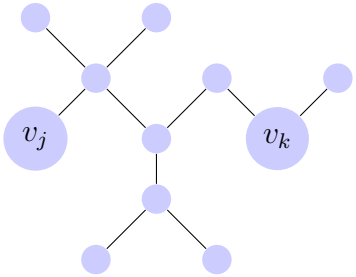


FIGURE 5. A possible structure for I_i. By fixing x_j, we can find x_k in terms of x_j.

Since $\operatorname{codim} \bigcap h(I_i) \leq |E(I_i)|$ because the codimension cannot be greater than the number of hyperplanes of $h(I_i)$, therefore $\operatorname{codim} h(I_i) = |V(I_i)| - 1$.

Now, since each $h(R_i)$ and each $h(I_i)$ uses a different set of coordinates, we can write $\operatorname{rk}(I)$ as $\sum_i \operatorname{rk}(R_i) + \sum_i \operatorname{rk}(I_i)$ (see the previous lemma about direct sums). Using the formulas we have just constructed, this becomes $\sum_i |V(R_i)| + \sum_i (|V(I_i)| - 1) = |V(I)| - b(I)$. Each vertex is included in either some R_i or some I_i, so this is $n - b(I)$.

For the proof of Part (1) to be complete, it remains to be shown that the vertex sets of the balanced components of I are those of the balanced components of S; in other words, that I restricted to a balanced component of S is connected. □

Lemma 31. *Suppose I is a maximal independent set in $S \subseteq E$. Let S have unbalanced components $U_1, U_2, \ldots$ and balanced components $B_1, B_2, \ldots$. Then $I \cap B_i$ is a tree spanning $V(B_i)$ and $I \cap U_i$ is a disjoint union of contrabalanced 1-trees.*

Proof. To appear next time. □

Lecture 5: The Frame Matroid, continued

13 November 2019
Notetaker: Nicholas Lacasse

We ended the last lecture with the statement of Lemma 31, of which the following is the essence,. We begin with proving this essence.

Lemma 32. *Let Φ be a $K^\times$-gain graph, $S \subseteq E(\Phi)$, and I a maximal independent set contained in S. Then $b(I) = b(S)$.*

Proof. Suppose S has balanced components $B_1, B_2, \ldots$ and unbalanced components $U_1, U_2, \ldots$. To prove that I and S have the same number of balanced components we will show two things:

(1) $I \cap E(B_j)$ is a spanning tree of B_j (and therefore balanced),
(2) $I \cap E(U_j)$ is a contrabalanced spanning 1-forest (and therefore has no balanced components).

For (1), notice that $I \cap B_j$ is a forest. This is because B_j is balanced, $I \subseteq B$, and I is independent. (If $I \cap B_j$ were not a forest there would be a balanced circle contained in I, but we know balanced circles are dependent. So $I \cap B_j$ is a forest.) Suppose it is disconnected. Then we can use an edge of B_j to join two trees of $I \cap B_j$ into one tree, since B_j is connected. Therefore, $I \cap B_j$ was not maximal, and thus I was not maximal. But we chose I to be maximal, so this is a contradiction. Thus $I \cap B_j$ is a tree, which spans B_j because an isolated vertex is a (very small) tree, so it falls under the previous argument.

For (2), notice that the only possible balanced components of $I \cap U_j$ are trees. By way of contradiction, suppose $I \cap U_j$ has a tree component, T. If T is not the only component of $I \cap U_j$ then we can add an edge of U_j to I in order to connect T to another component U of $I \cap U_j$ (this other component of $I \cap U_j$ will necessarily be either a tree, a tree with a single half-edge, or an unbalanced unicycle because I is a contrabalanced pseudoforest), giving a larger independent set (because connecting T to a component which is either a tree, a tree with a single half-edge, or an unbalanced unicycle$U$ will just yield a larger component of the same type as U, preserving the contrabalanced pseudoforest). Since we chose I to be maximally independent, this is a contradiction. So T must be the only component of $I \cap U_j$. Then either U_j has a half-edge e, in which case $T \cup e$ is a larger independent set, or U_j has an unbalanced circle C, which can be extended to a unicycle U that spans U_j (since U_j is connected). We can replace T by U to create an independent set larger than I. This is a contradiction because maximal independent sets all have the same size (by matroid theory). So $I \cap U_j$ has no tree components. Therefore $I \cap U_j$ has no balanced components. So in I, we get exactly one balanced component for each B_j and no others. Therefore, $b(I) = b(S)$. $\square$

N.B. Once we have established the independent sets of the frame matroid $\mathbf{F}(\Phi)$ using $\mathscr{M}(\mathscr{H}[\Phi])$, we can define $\mathrm{rk}(S) = \max\{\#I \mid I \subseteq S,\ I \text{ independent}\}$ for $S \subseteq E(\Phi)$ entirely in terms of the gain graph. We never need to refer back to $\mathscr{H}[\Phi]$. This permits a vast generalization. Take any biased graph $(\Gamma, \mathscr{B})$, possibly from a gain graph and possibly not, and define $S \subseteq E$ to be independent if it is a contrabalanced pseudoforest.

Theorem 33. *Define $\mathbf{F}(\Gamma, \mathscr{B}) := ((\Gamma, \mathscr{B}), \mathscr{I})$ where $\mathscr{I}$ is the set of all $S \subseteq E(\Gamma)$ which induce contrabalanced pseudoforests. Then $\mathbf{F}(\Gamma, \mathscr{B})$ is a matroid on E.*

Proof. The proof will appear later. We have not even proved this theorem for a $\mathfrak{G}$-gain graph when $\mathfrak{G}$ is not the multiplicative group of a field! But it is true. □

Given Theorem 33, our proof that $\operatorname{rk}(S) = n - b(S)$ for $S \subseteq E$ and our proof of the circuits of $\mathbf{F}(\Phi)$ both apply to $\mathbf{F}(\Gamma, \mathscr{B})$.

The next step in the proof of Theorem 25 is to establish the closure operator.

Lemma 34. $\operatorname{cl}_{\mathbf{F}}(S) = (E{:}V_0(S)) \cup \operatorname{bcl}(S{:}V_0(S)^c)$.

For this we will prove a helpful lemma.

Lemma 35. *Let Φ be a gain graph and let $R \subseteq E$ be balanced and connected. Then* $\operatorname{bcl}(R)$ *(aside from the balanced loops and loose edges) is the maximal balanced subset of E that contains R and has vertex set contained in $V(R)$.*

Proof. Since R is balanced we can switch Φ so it has all identity gain. Then $\operatorname{bcl}(R) = \{e \in E{:}V(R) \mid \varphi(e) = \varepsilon\}$ together with all balanced loops and loose edges. □

For an arbitrary edge set R we have a reduction to components, which is of most use when R is balanced.

Lemma 36. *Let Φ be a gain graph and $R \subseteq E$. Let* $\operatorname{Comp}(R)$ *be the set of components of R. Then* $\operatorname{bcl}(R) = \bigcup_{C \in \operatorname{Comp}(R)} \operatorname{bcl}(C)$.

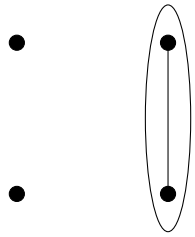
Proof. An exercise. □

Here we see that balance-closure differs from closure in that bcl never joins unbalanced components of R, but cl can join them.

Lecture 6: Closure and Closed Sets

15 November 2019
Notetaker: Andrew Lamoureux

Let's recall some definitions from graph theory. Let $\Gamma = (V, E)$ be a graph and $S \subseteq E$. (We always use V and E for the edge sets of the graph Γ.) The vertex set of S is $V(S) := \{v \in V \mid \exists\ e \in S \text{ such that } v \text{ is an endpoint of } e\}$. Then for any $X \subseteq V$ satisfying $X \supseteq V(S)$, (X, S) is also a graph. (This is how every subgraph of Γ is formed.) For example, in the graph Γ below, V consists of all four vertices. If S is the singleton whose element is the edge, then the elements of $V(S)$ are the two vertices within the ellipse.



Moving over to a gain graph (or biased graph), V and E always denote its vertex and edge set.

As for "spanning", we are using it for two incompatible notions. (Not our fault; it's "Tradition!".) In graph theory, a *spanning subgraph* of a graph Γ is a subgraph whose vertex set is all of V. In matroid theory, a *spanning set* of a matroid is a subset of the ground set whose closure is the entire ground set. When an edge set S is said to be spanning, it is spanning in the matroid sense, because we are working with matroids on the edge set (and because an edge set is not a subgraph).

Finally, let's recall a definition from matroid theory. Let M be a matroid with ground set E. The *closure* of $S \subseteq E$ is $\mathrm{cl}(S) := S \cup \{e \in E \mid \mathrm{rk}(S \cup \{e\}) = \mathrm{rk}(S)\}$. We will be proving a more graphic formula in the next theorem. *N.B.* By matroid theory, since loose edges and balanced loops have rank 0, they are in every closed edge set.

Additional notation: We denote the number of balanced components of any subgraph (X, S) of a gain or biased graph, where $X \subseteq V$ and $S \subseteq E$ (necessarily with $V(S) \subseteq X$), by $b(X, S)$. Thus, for instance, our usual notation $b(S)$ is shorthand for $b(V, S)$.

Theorem 37 (Theorem 25(3))**.** *Let $\Phi = (\Gamma, \varphi)$ be a gain graph with edge set E and $S \subseteq E$. Then*

$$\mathrm{cl}(S) = (E{:}V_0(S)) \cup \mathrm{bcl}(S{:}V_0(S)^c).$$

(All the loose edges and balanced loops are included in $\mathrm{bcl}(S{:}V_0(S)^c)$.) This follows from a lemma:

Lemma 38. *The balance-closure*[1] *of a balanced edge set B is*

$$\mathrm{bcl}(B) = B \cup \{e \in E{:}V(B) \mid B \cup \{e\} \textit{ is balanced}\} \cup \{\textit{loose edges and balanced loops}\}$$

Proof. Recall that $\mathrm{bcl}(B) = B \cup \{e \in E(\Gamma) \mid \exists \text{ a path } P \text{ in } B \text{ joining endpoints of } e \text{ such that } P \cup \{e\} \text{ is balanced}\} \cup \{\text{loose edges}\}$. Let A be the set in the statement of the lemma. As $\mathrm{bcl}(B)$ is balanced by Lemma 30, $\mathrm{bcl}(B) \subseteq A$.

[1] Reminder: Not "balanced closure".

Conversely, let $e \in (E{:}V(B)) \setminus B$. Then the endpoints of e are joined by a path P in B. But $B \cup \{e\}$ is balanced, as $e \in A$, so $P \cup \{e\}$ is a balanced circle, and $e \in \mathrm{bcl}(B)$. □

Proof of Theorem 37. Recall that $\mathrm{rk}(S) = n - b(S)$, so $\mathrm{cl}(S) = S \cup \{e \in E \mid b(S \cup \{e\}) = b(S)\}$. Clearly, no edge of $E : V_0(S)$ increases $b(S)$ when added to S: the balanced components remain the same. It's also clear that $\mathrm{cl}(S)$ can't contain any edge that connects a balanced component of S to any other component, as that would reduce $b(S)$.

The only remaining case is an edge whose endpoints are in a balanced component B. Again by Lemma 30, $\mathrm{bcl}(B)$ is balanced, so replacing B by $\mathrm{bcl}(B)$ preserves $b(S)$ and makes S larger (or leaves it unchanged).

Suppose we add an edge $e \in (E : V(B)) \setminus \mathrm{bcl}(B)$. By the above lemma, $B \cup \{e\}$ is not balanced. This reduces $b(S)$ by 1, so $e \notin \mathrm{cl}(S)$. □

Theorem 39 (Theorem 25(4))**.** *A set $S \subseteq E$ is closed if and only if*

$$S = (E{:}V_0(S)) \cup \mathrm{bcl}(S{:}V_0(S)^c) \cup \{\textit{loose edges and balanced loops}\}.$$

Proof. By definition, S is closed iff $S = \mathrm{cl}(S)$. □

The preceding description is equivalent to the following: S is closed iff its unbalanced part is an induced edge set, each balanced component is balance-closed, and all loose edges and balanced loops of Φ are in S. We get all closed sets by taking all sets of the form

$$(E{:}X) \cup B \cup \{\text{loose edges and balanced loops}\}$$

where $X \subseteq V$, $E{:}X$ has no balanced components, and $B \subseteq E{:}X^c$ is balanced and balance-closed.

Graphs vs. gain graphs vs. biased graphs. The notion of biased graph is not exactly a generalization of that of a gain graph. Rather, it is an *abstraction* of a gain graph, because a biased graph does not have any gains; it retains only the combinatorial structure of a gain graph.

Similarly, a gain graph is not a generalization of a graph but a graph *with additional structure*. While it is true that any graph Γ can be made a $\mathfrak{G}$-gain graph for any group $\mathfrak{G}$ by declaring $\varphi(e) = \varepsilon$ (the identity) for all $e \in E(\Gamma)$, this is not always the "right" choice. It is "right" in the following sense: the hyperplane arrangement of Γ without a gain is exactly the same as the arrangement of Γ with all-identity gains. However, for a signed graph, which is a $\mathbb{Z}_2$-gain graph, in some generalizations of ordinary graph theory (e.g., in regard to line graphs) it is preferable to assign every edge a negative sign, i.e., gain -1 rather than 1.

Gains vs. weights; orientation vs. direction. Let's clarify two subtle distinctions.

A *gain* is not the same as a *weight*. A gain is inverted when using the opposite orientation, while a weight is not (there need not even be orientation). (This is my personal distinction. It seems to be consistent with general usage, although it is not rigorously followed.)

The difference between an orientation, for the purpose of defining the gain, and no orientation shows up clearly in directed graphs. In a directed graph with gains, or *gain digraph*, each edge has only one direction. For example, a path or any walk must follow the directions of the edges; therefore, not every circle can be said to have a gain, but only the ones that are consistently directed.

Lecture 7: The Chromatic Polynomial and the Balanced Chromatic Polynomial

20 November 2019
Notetaker: Michael Gottstein

Recall that for a graph $\Gamma = (V, E)$, the chromatic polynomial of Γ is

$$\chi_\Gamma(\lambda) = \sum_{S \subseteq E} (-1)^{\#S} \lambda^{c(S)} = \sum_{A \in \operatorname{Lat} \Gamma} \mu(\varnothing, A) \lambda^{c(A)},$$

where $c(S) = c(V, S)$ is the number of components of (V, S) and $\operatorname{Lat} \Gamma$ is the lattice of flats of the graphic matroid of Γ. The definition in a biased graph, including a gain graph, is very similar; there is only one big little difference.

Definition 40. The *chromatic polynomial* of a biased graph $\Omega = (\Gamma, \mathscr{B})$, where E is the edge set of Γ, is

$$\chi_\Omega(\lambda) := \sum_{S \subseteq E} (-1)^{\#S} \lambda^{b(S)} = \sum_{A \in \operatorname{Lat} \Omega} \mu(\varnothing, A) \lambda^{b(A)}.$$

The second equality follows from the Möbius function formula for matroids mentioned in Stanley's notes.

If S is balanced then $b(S) = c(S)$, so if Ω is balanced the chromatic polynomial of the gain graph agrees with the chromatic polynomial of the underlying graph.

Recall our definition that, if the empty set is not closed, then $\mu(\varnothing, A) = 0$ for every flat A. Consequently, if Ω contains a loose edge or a balanced loop, then its chromatic polynomial is identically 0.

Definition 41. A biased graph has a second chromatic polynomial. First we have to define the *meet semilattice of balanced flats*,

$$\operatorname{Lat}^{\mathrm{b}}(\Omega) := \{A \in \operatorname{Lat} \Omega : A \text{ is balanced}\}.$$

The *balanced chromatic polynomial* of Ω is

$$\chi^{\mathrm{b}}_\Omega(\lambda) := \sum_{S \subseteq E : \text{balanced}} (-1)^{\#S} \lambda^{b(S)} = \sum_{A \in \operatorname{Lat}^{\mathrm{b}} \Omega} \mu(\varnothing, A) \lambda^{b(A)}.$$

If Ω is balanced, the balanced chromatic polynomial, like the chromatic polynomial, is the same as the chromatic polynomial of Γ. In other words, all three coincide. However, if Ω is not balanced, all three are different. (I will not prove that, but you can see that the range of summation for χ^{b}_Ω is smaller than that for χ_Ω and χ_Γ, and the exponents in the two latter sums are unequal for unbalanced sets S.

We now come to a nice generalization of the theorem about the characteristic polynomial of a graphic hyperplane arrangement (in Stanley's notes).

Theorem 42. *For a $K^\times$-gain graph Φ, $p_{\mathscr{H}[\Phi[}(\lambda) = \chi_\Phi(\lambda)$.*

Proof. This follows from the rank formula $\operatorname{rk}(S) = n - b(S)$ and the known s for the characteristic polynomial of a hyperplane arrangement. In $p_{\mathscr{H}[\Phi[}(\lambda)$, the exponent $\dim h(S) = n - \operatorname{rk} h(S) = b(S)$ because $\operatorname{rk} h(S) = \operatorname{rk}_\Omega(S) = n - b(S)$. (Recall that for $S \subseteq E$, $h(S)$ is the set of corresponding hyperplanes and that the frame matroid $\mathbf{F}(\Phi)$ is isomorphic by h to $\mathscr{M}(\mathscr{H}[\Phi])$.) $\square$

Now that we have a polynomial defined on a gain graph. let's see what it can do.

Definition 43. Given a gain graph Φ we define a *k-coloration* as a function

$$\gamma : V(\Phi) \to \mathfrak{G} \times [k] \cup \{0\}.$$

A *zero-free k-coloration* is a coloration that does not use 0; in other words, it is a function

$$\gamma : V(\Phi) \to \mathfrak{G} \times [k].$$

We call the codomain the *color set* and denote it by $\mathbf{C}_k^0(\mathfrak{G})$, or $\mathbf{C}_k(\mathfrak{G})$ when we do not include 0.

Define a right action of $\mathfrak{G}$ on $\mathbf{C}_k^0$ and thus on $\mathbf{C}_k$ by $0g := 0$ and $(h, i)g := (hg, i)$ for $g, h \in \mathfrak{G}$ and $i \in [k]$. A coloration is *proper* if, for every ordinary edge e_{vw}, $\gamma(w) \neq \gamma(v)\varphi(e_{v,w})$, and also $\gamma(v) \neq 0$ for each vertex v that supports a half edge or an unbalanced loop.

We come at last to the main result of today's lecture.

Theorem 44 (Proper Coloration)**.** *If $\mathfrak{G}$ is finite, say of order m, then $\chi_\Phi(km + 1)$ is the number of proper k-colorations of Φ and $\chi_\Phi^{\mathrm{b}}(km)$ is the number of zero-free proper k-colorations.*

Observe the interesting fact that the number of proper colorations is independent of the particular group. It depends only on how big the group is.

Example 45 (Gain Graphs vs. Hyperplane Arrangements)**.** Let $\mathfrak{G} = K^\times$ and $k = 1$. Then $\mathbf{C}_1 = K$ (as a multiplicative monoid) and $\mathbf{C}_1^0 = K^\times$.

We can generalize our choice of color set so as to be able to count proper colorations of the gain graph of a gain-graphic hyperplane arrangement over an infinite field. Let $\mathfrak{G}$ be any finite subgroup of $K^\times$. Examples are a finite cyclic group of any order as a subgroup of $\mathbb{C}^\times$ and the multiplicative group of a finite field $\mathbb{F}_q$, which is a cyclic group of order $q - 1$ (q being a prime power).

Problem 46 (Other Evaluations)**.** We only gave interpretations for nonnegative $\lambda \equiv 0, 1 \pmod m$, where $m = \#\mathfrak{G}$. As far as I know, there are no combinatorial interpretations of other nonnegative values of λ. There is a potential research problem.

Interpretations of negative values other than -1 are also a mystery.

The sole exception to the mystery is when $m = 2$, i.e., signed graphs. There might be hints there of generalizations ...

Example 47 (Signed Graphs)**.** The special case of a group of order 2 is exceptionally interesting and more studied. Let $\mathfrak{G} = \{\pm 1\} \subseteq K^\times$, where $\operatorname{char} K \neq 2$ (that is, $1 \neq -1$). We call such a gain graph Φ a *signed graph* [3]. We can treat the color set as a sign-symmetric set of integers if we color with $\{0, \pm 1, \pm 2, \ldots, \pm k\}$—as long as $\operatorname{char} K$ is large enough (the colors must be distinct), and in particular whenever $\operatorname{char} K = 0$.

In signed graphs, $\chi_\Phi(2k + 1)$ gives us the number of proper k-colorations when λ is odd, provided we set $k = \frac{1}{2}(\lambda - 1)$; and $\chi_\Phi^{\mathrm{b}}(2k)$ gives us the number of zero-free proper k-colorations when λ even, if we take $k = \frac{1}{2}\lambda$. And more: negative evaluations count "compatible pairs" of orientations and colorations (generalizing a deep theorem of Stanley's), but this is outside our scope.

Even more: the existence of two polynomials is explained by the Ehrhart theory of lattice-point counting in convex polytopes with fractional vertices [1, §5]—but that, too, is outside

our scope. No such explanation exists for larger gain groups. I wonder if finite cyclic groups might be understood better by way of complex hyperplane arrangements, since such a group is a subgroup of $\mathbb{C}^\times$.

There is extensive literature on signed graphs, though not much on their coloring (mainly [11, 12, 4]). Much of it is not in mathematics, but is directed towards social science, inspired by a foundational article of Cartwright and Harary from 1956 [2]. On the other hand, the hyperplane arrangements of signed graphs are implicitly connected with the major mathematical topic of Lie theory via root systems (*q.v.*), which are becoming important in combinatorial geometry.

Definition 48. Sometimes we only want a 0-free 1-coloration; we call that a *group coloration* as it simply maps $V \to \mathfrak{G}$ (notationally simplified from $\mathfrak{G} \times [1]$).

Sometimes we like to regard the set $[k]$ as the group $\mathbb{Z}_k$ and view a k-coloration as a group coloration in the cyclic group (independently of what the gain group is), much as $k = 1$ gives a group coloration in the gain group.

Lecture 8: Deletion, Contraction, and Coloration

10 November 2019
Notetaker: Jimmy West

The main topic of this lecture is coloring gain graphs, but for the principal theorems we have to define not only deletion of edges, which is obvious, but also contraction of edges, which is far from obvious for gain graphs, in contrast to how it is for ordinary graphs.

Deletion and Contraction.
We begin with gain graphs, then apply our ideas to biased graphs.

Definition 49 (Deletion of an Edge)**.** Deletion of an edge in a gain graph or a biased graph is trivial. It should be noted that all gains remain the same upon deletion and the balanced circles remain the same, except for those that are no longer circles upon the deletion.

Definition 50 (Contraction of an Edge in a Gain Graph)**.** The notation for a gain graph Φ with e contracted is Φ/e.

To contract a link e in Φ, first switch Φ so e has gain ε, then coalesce the endpoints, and finally delete the contracted edge e.

To contract a loose edge or a balanced loop, simply delete the edge. Do not change the gains of other edges.

To contract a half edge or an unbalanced loop e incident with vertex v, remove both v and e but not any other edges. This may remove some endpoints of some edges; in particular, it reduces a link e that joins v to w to a half edge incident with w, and a loop or half edge at v (other than e itself) to a loose edge. Do not change the gains of edges that remain ordinary edges—but an ordinary edge that becomes a half or loose edge no longer has a gain.

Many different switching functions can give e the switched gain ε, so the contraction Φ/e is not uniquely defined. All different contractions are switching equivalent.

Proposition 51. *Suppose switchings Φ^ζ and $\Phi^{\zeta'}$ both give gain ε to the link e. The resulting contractions Φ^ζ/e and $\Phi^{\zeta'}/e$ are switching equivalent.*

Proof. A good exercise about switching. □

In other words, contraction is well defined on switching classes (cf. Definition 10). More precisely, the contraction $[\Phi]/e$ of a switching class is a uniquely determined switching class of contracted graphs, which may naturally also be notated as $[\Phi/e]$. This suggests that contraction really acts on switching classes and that switching classes are more fundamental than signed graphs, but I will not pursue that line of thought here.

Since a biased graph has no gains, the definition of contraction has to be adapted, and in such a way that it is compatible with contraction in a gain graph.

Definition 52 (Contraction of an Edge in a Biased Graph)**.** For a biased graph $\Omega = (\Gamma, \mathscr{B})$, again there are different rules for different kinds of edge. The notation for Ω with e contracted is Ω/e.

To contract a link e, we contract it in the underlying graph Γ. Then we have to define the bias. A circle D in Ω/e is balanced if it is the contraction of a balanced circle C in Ω that contains e, or if it is a circle in Ω that is balanced. Otherwise, it is unbalanced; that is,

if it is the contraction of anun balanced circle in Ω that contains e, or if it is an unbalanced circle in Ω.

To contract a loose edge or a balanced loop, simply delete the edge.

To contract a half edge or an unbalanced loop e incident with vertex v, remove both v and e but not any other edges. This may remove some endpoints of some edges; in particular, it reduces a link e that joins v to w to a half edge incident with w, and a loop or half edge at v (other than e itself) to a loose edge.

It is worthwhile to point out why this is a complete definition of the balanced circle class in Ω/e. Suppose we contract a link e. A circle C in Ω that contains e will contract to a circle with edge set $C \setminus e$ in Γ/e, and whether or not it is balanced will not be affected by contraction. If C does not contain e, there are two cases. If both endpoints of e, say v and w, are in C, then C contracts to a pair of circles, each consisting of one of the two vw-paths in C. Otherwise, C simply remains a circle in the contracted graph.

There is one little difficulty. Unlike with gain graphs, where it is obvious, how do we know the contracted biased graph is a biased graph?

Proposition 53. *If Ω is a biased graph and e is an edge of Ω, then Ω/e is a biased graph.*

Proof. This is an excellent exercise for the reader. □

Given a gain graph Φ, we denote the corresponding biased graph by $\langle\Phi\rangle := (\Gamma, \mathscr{B}(\Phi))$. It should be clear that, if we have a gain graph Φ and contract an edge e, then take the biased graph of Φ/e, we get the same result as if we contract e after taking the biased graph of Φ. Symbolically,

$$\langle\Phi/e\rangle = \langle\Phi\rangle/e.$$

Example 54. We do an example of deletion and contraction using the $\mathbb{R}^\times$-gain graph Φ in Figure 6.

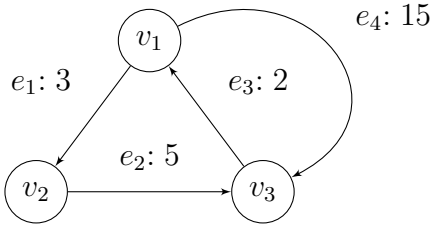


FIGURE 6. A gain graph Φ, with the gains listed on each edge.

In the example we contract the link e_4, giving the graph Φ/e_4. First, we switch the gains so that e_4's gain is the identity. In our example we will use the switching function defined by:

$$\zeta(v_1) = 1, \quad \zeta(v_2) = 1, \quad \zeta(v_3) = \frac{1}{15}.$$

The resulting gains are then

$$\varphi^\zeta(e_1) = \zeta(v_1)^{-1}\varphi(e_1)\zeta(v_2) = 1 \times 3 \times 1 = 3,$$
$$\varphi^\zeta(e_2) = \zeta(v_2)^{-1}\varphi(e_3)\zeta(v_3) = 1 \times 5 \times \frac{1}{15} = \frac{1}{3},$$
$$\varphi^\zeta(e_3) = \zeta(v_3)^{-1}\varphi(e_3)\zeta(v_1) = 15 \times 2 \times 1 = 30,$$
$$\varphi^\zeta(e_4) = \zeta(v_1)^{-1}\varphi(e_4)\zeta(v_3) = 1 \times 15 \times \frac{1}{15} = 1.$$

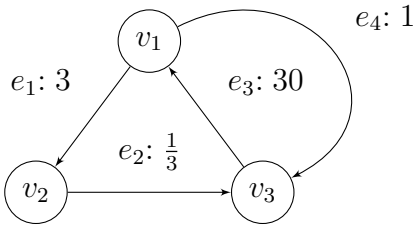


FIGURE 7. The gain graph Φ^ζ, with the switched gains.

For the second step, contract e_4, keeping all gains. The resulting gain graph, Φ/e_4, is shown in Figure 8. Notice that the digon in Φ/e_4 is balanced and that these edges came from the

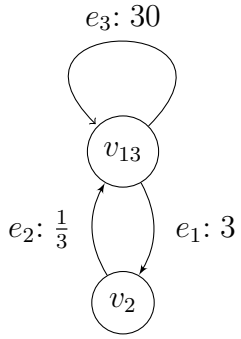


FIGURE 8. Φ/e_4; The vertex v_{13} is the coalescence of vertices v_1 and v_3.

balanced circle $e_1e_2e_4^{-1}$ in Φ. In fact, when contracting edge e, every circle C through e becomes a new circle C/e that is balanced if, and only if, C was originally balanced. This is because the gains of circles do not change under switching and the gains of edges do not change when e is contracted.

Example 55. Next, we look at contraction of an unbalanced loop (or a half edge). Let $\Psi = \Phi/e_4$ from Example 54 and consider Ψ/e_3. The two endpoints of this loop coincide and the loop is unbalanced. We delete the edge and remove its incident vertex v_{13}. All edges incident with v_{13} lose that vertex; thus, e_1 and e_2 become half edges. The contraction Ψ/e_3 is shown in Figure 9.

There are no edges not incident with v_{13}, but if there were, they would retain their gains. If there were any half edges or loops incident to the deleted vertex, they would become loose edges; see Figure 10.

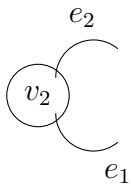


FIGURE 9. Ψ/e_3; Since the only remaining edges are half edges, there are no longer any gains.

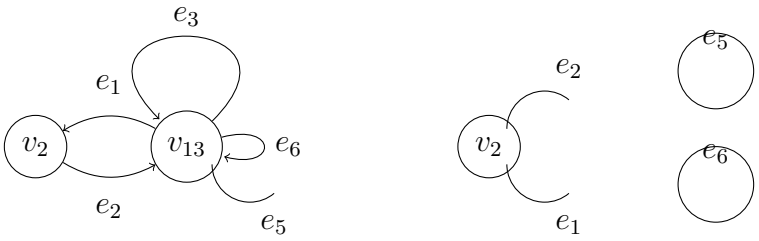


FIGURE 10. The left graph Ψ' is Ψ with two added edges. The gains of the added edges are irrelevant to the example. On the right is the contraction Ψ'/e_3, with two half edges and two loose edges. Those edges no longer have gains.

Chromatic polynomials.

Now we can return to chromatic polynomials.

An edge is said to be balanced if the edge, as an edge set, is balanced. That is, a link, a balanced loop, and a loose edge are balanced, while a half edge and an unbalanced loop are unbalanced.

Observe that, by their algebraic definitions (and writing $\Omega = \langle\Phi\rangle$ for brevity), $\chi_\Phi = \chi_\Omega$ and $\chi^{\mathrm{b}}_\Phi = \chi^{\mathrm{b}}_\Omega$.

Theorem 56 (Deletion-Contraction Formula for Chromatic Polynomials)**.** *For a gain graph Φ and an edge e,*

$$\chi_\Phi(\lambda) = \chi_{\Phi\setminus e}(\lambda) - \chi_{\Phi/e}(\lambda)$$

and

$$\chi^{\mathrm{b}}_\Phi(\lambda) = \begin{cases} \chi^{\mathrm{b}}_{\Phi\setminus e}(\lambda) - \chi^{\mathrm{b}}_{\Phi/e}(\lambda) & \text{if } e \text{ is a balanced edge,} \\ \chi^{\mathrm{b}}_{\Phi\setminus e}(\lambda) & \text{if } e \text{ is an unbalanced edge.} \end{cases}$$

The same is true for a biased graph Ω.

We will prove the theorem for biased graphs. The result for gain graphs follows directly because the polynomials are the same.

Proof for the chromatic polynomial. The definition says

$$\chi_\Omega(\lambda) = \sum_{S\subseteq E} (-1)^{\#S} \lambda^{b(S)}.$$

We break the sum up into two parts, one for the edge sets that contain e and one for those that do not. Thus,

$$\chi_\Omega(\lambda) = \sum_{S \subseteq E \setminus e} (-1)^{\#S} \gamma^{b(S)} + \sum_{e \in S \subseteq E} (-1)^{\#S} \gamma^{b(S)}.$$

The first sum is $\chi_{\Omega \setminus e}(\lambda)$. For the second sum, we write each edge set S containing e as $T \cup e$ for some edge set $T \subseteq E \setminus e$. The sum then becomes $\sum_{T \subseteq E \setminus e} (-1)^{\#T+1} \lambda^{b_\Omega(T \cup e)}$. Note that the exponent is the number of balanced components in Ω, not in Ω/e. We now have

$$\chi_\Omega(\lambda) = \chi_{\Omega \setminus e}(\lambda) - \sum_{T \subseteq E/e} (-1)^{\#T} \lambda^{b_\Omega(T \cup e)}$$

The sum over T equals $\chi_{\Omega/e}(\lambda)$ by the two following lemmas, which complete the proof of Theorem 56. □

Lemma 57. *If Ω is balanced and $e \in E$, then Ω/e is also balanced.*
Conversely, if e is balanced in Ω and Ω/e is balanced, then Ω is balanced.

Proof. If C is a circle in Ω/e, then either (1) $C \cup e$ is a circle in Ω, which is balanced since Ω is balanced, so C is balanced in Ω/e, or (2) C is itself a circle in Ω, and it is balanced in Ω/e. Further, a balanced graph does not contain any half edge or unbalanced loop, so after contraction of an edge, it will have no half edge. That proves the first half of the lemma.

Conversely, if e is a link in Ω (the other cases being trivial), there can be no half edges in Ω, so if Ω is unbalanced, there is an unbalanced circle C that contains e. But then C/e would be an unbalanced circle in Ω/e, contrary to hypothesis. That establishes the second half of the lemma. □

Lemma 58. *$b_\Omega(T \cup e) = b_{\Omega/e}(T)$ for any edge e and any set $T \subseteq E \setminus e$.*

Proof. If e is in a balanced component, we use Lemma 57.

If e is in an unbalanced component, say U, we have three cases.

Case 1. e is a link. If e is in an unbalanced circle C, then C/e is an unbalanced circle in U/e. Hence U/e is unbalanced. Suppose the component has an unbalanced circle C such that C is a circle in U/e. Then C is unbalanced in U/e, so U/e is unbalanced. Suppose U has an unbalanced circle C of which e is a chord. Then $C \cup e$ contains two other circles, say C_1 and C_2. At least one of these must be unbalanced, say C_1. Then C_1/e is unbalanced in U/e. Therefore U/e is unbalanced in this case as well. Hence $b(\Omega/e) = b(\Omega)$.

Case 2. if e is a balanced loop or a loose edge, then $\Omega/e = \Omega \setminus e$, so $b(S \cup e) = b(S)$.

Case 3. If e is a half edge or unbalanced loop at a vertex v, then e is in an unbalanced component $T \cup e$. Every component of $T \setminus v$ has at least one edge joining it to v. Thus, by the definition of contraction every component of $(T \cup e)/e$ will have a half edge so it is unbalanced. Thus, the balanced components are the same as before contraction.

That completes the proof of Lemma 58. □

Lecture 9: Chromatic Polynomials Count Colorations

22 November 2019
Notetaker: Shuchen Mu

Proof of Theorem 56 for the balanced chromatic polynomial. The definition is

$$\begin{aligned}\chi^{\mathrm{b}}_{\Phi}(\lambda) &= \sum_{\substack{S\subseteq E\\ S\text{balanced}}} (-1)^{\#S}\lambda^{b(S)} \\ &= \sum_{\substack{S\subseteq E\setminus e\\ S \text{ balanced}}} (-1)^{\#S}\lambda^{b(S)} - \sum_{\substack{T\subseteq E\setminus e\\ T\cup e \text{ balanced}}} (-1)^{\#T}\lambda^{b_\Omega(T\cup e)}.\end{aligned}$$

By Lemma 58, the last summation equals

$$\sum_{\substack{T\subseteq E\setminus e\\ T\cup e \text{ balanced}}} (-1)^{\#T}\lambda^{b_{\Omega/e}(T)}$$

Since e is a balanced edge, by Lemma 57 the edge set T of Ω/e is balanced if and only if $T\cup e$ is balanced in Ω. Hence, the last summation

$$= \sum_{\substack{T\subseteq E(\Omega/e)\\ T \text{ balanced}}} (-1)^{\#T}\lambda^{b_{\Omega/e}(T)}.$$

That concludes the proof for a link.

For a balanced loop or a loose edge, boht left side and right side equal 0.

For an unbalanced edge, deleting the edge does not affect the count of zero-free proper colorations. □

Switching of colorations. We define the switching γ^ζ of a coloration γ with respect to a switching function ζ to be

$$\gamma^\zeta(v) = \gamma(v)\zeta(v).$$

We see that

$$\begin{aligned}\gamma(v)\zeta(v)(\zeta(v)^{-1}\varphi(e_{vw})\zeta(w)) = \gamma(v)\varphi(e_{vw})\zeta(w) = \gamma(w)\zeta(w)\\ \iff \gamma(w) = \gamma(v)\varphi(e_{vw}).\end{aligned} \tag{1}$$

Therefore a coloration is proper at a link if and only it is proper at the link after switching.

Define

$$K(\Phi) := \text{the set of proper } k\text{-colorations of } \Phi,$$

and similarly $K(\Phi\setminus e)$ = the set of proper k-colorations of $\Phi\setminus e$ and $K(\Phi/e)$ = the set of proper k-colorations of Φ/e. We are now obliged to mention something that we swept under the rug in defining contraction: the gains of Φ/e depend on the choice of the switching function ζ by which we switched e to have gain ε. Nevertheless, all possible contractions Φ/e are switching equivalent (that is an exercise for the reader), so it follows from Equation (1) that, although Φ/e depends on the choice of switching function, the resulting $K(\Phi/e)$'s are all bijective to each other by bijections that preserve the 0-colored set, and therefore their cardinalities are all the same.

Counting proper colorations. At last we can prove the main theorem about the chromatic polynomials.

Theorem 59. *Assume $\mathfrak{G}$ is finite and Φ is a $\mathfrak{G}$-gain graph of finite order. Let $m = |\mathfrak{G}|$. Then*

$$\chi_\Phi(km+1) = \textit{the number of proper } k\textit{-colorations of } \Phi, \textit{ and}$$
$$\chi^{\mathrm{b}}_\Phi(km) = \textit{the number of zero-free proper } k\textit{-colorations.}$$

We formulate the main parts of the proof as two lemmas. The theorem will follow by some special cases and induction on the number of edges.

For the first part of the theorem we define $\hat{\chi}_\Phi(km+1) :=$ the number of proper k-colorations of Φ. So $\hat{\chi}_\Phi$ is a function evaluated at positive integers of residue 1 (mod m).

Lemma 60. *If e is a link in Φ, then $\hat{\chi}_\Phi = \hat{\chi}_{\Phi\setminus e} - \hat{\chi}_{\Phi/e}$.*

Proof. Let e_{vw} be a link in Φ, also denoted more briefly by e. We simplify the proof by assuming Φ has been switched so e has gain ε. Then contraction of e does not require any switching.

A coloration is proper if and only if it is proper at each edge, i.e., $\gamma(b) \neq \gamma(a)\varphi(e_{ab})$ for every edge e_{ab}. Since $\Phi \setminus e$ has all the vertices and edges of Φ except e,

$$K(\Phi) = \{\gamma \in K(\Phi \setminus e) \mid \gamma(w) \neq \gamma(v)\varphi(e_{vw}) = \gamma(v)\}.$$

Consider a coloration that is proper except at e_{vw}. It gives a proper coloration of Φ/e because the color at v is the same as at w. In Φ/e, the color of the contraction vertex v_e is $\gamma(v)$; all other vertices retain the same color as in Φ.

Contrariwise, given a proper coloration $\hat{\gamma}$ of Φ/e, we construct a coloration γ of Φ by

$$\gamma(u) = \begin{cases} \hat{\gamma}(u) & \text{if } u \neq v, w, \\ \hat{\gamma}(v_e) & \text{if } u = v \text{ or } w, \end{cases}$$

where v_e is the contraction vertex. It is easy to see that these two constructions are inverse to each other, so they give a bijection between $\{\gamma \in K(\Phi \setminus e) \mid \gamma(w) = \gamma(v)\varphi(e_{vw})\}$ and $K(\Phi/e)$.

Hence we have proved for the case when e is a link that there is a bijection

$$K(\Phi) \uplus K(\Phi/e) \longleftrightarrow K(\Phi \setminus e).$$

It follows that $\hat{\chi}_\Phi = \hat{\chi}_{\Phi\setminus e} - \hat{\chi}_{\Phi/e}$. □

For the second part of the proof we define $\hat{\chi}^{\mathrm{b}}_\Phi(km) :=$ the number of zero free proper k-colorations of Φ. So $\hat{\chi}^{\mathrm{b}}_\Phi$ is a function evaluated at nonnegative integer multiples of m.

Lemma 61. *If e is a link in Φ, then $\hat{\chi}^{\mathrm{b}}_\Phi = \hat{\chi}^{\mathrm{b}}_{\Phi\setminus e} - \hat{\chi}^{\mathrm{b}}_{\Phi/e}$.*

We will prove this lemma in the next lecture.

Lecture 10: Examples!

25 November 2019
Notetaker: Nicholas Lacasse

In this lecture, we use the results we have developed to explore various examples.

Definition 62. Let $\mathfrak{G}$ be a group and Γ a simple graph. Then $\mathfrak{G}\Gamma = (V(\Gamma), \mathfrak{G} \times E(\Gamma))$ with $\varphi(g, e_{ij}) = g$ is a gain graph called a *group expansion* of Γ, specifically the $\mathfrak{G}$-*expansion*.

The *full* $\mathfrak{G}$-*expansion* $\mathfrak{G}\Gamma^\bullet$ is $\mathfrak{G}\Gamma$ with a half edge added at every vertex.

See Figure 11 for the $\mathfrak{G}$-expansion of a link e.

Theorem 63. *Let $\mathfrak{G}$ be a finite group of order m and Γ a simple graph on n vertices. Then*

$$\chi^{\mathrm{b}}_{\mathfrak{G}\Gamma}(\lambda) = m^n \chi_\Gamma\Big(\frac{\lambda}{m}\Big)$$

and

$$\chi_{\mathfrak{G}\Gamma^\bullet}(\lambda) = m^n \chi_\Gamma\Big(\frac{\lambda - 1}{m}\Big).$$

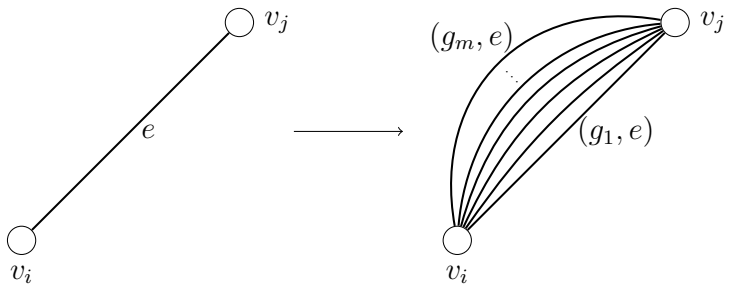


Figure 11. On the left is the edge e in Γ. On the right is the set of edges which places e in $\mathfrak{G}\Gamma$, one edge for each element of $\mathfrak{G}$.

Proof. For the first part of the theorem, we start with a 0-free k-coloration γ of $\mathfrak{G}\Gamma$. This coloration is proper when, for all $f \in \mathfrak{G}$, $\gamma(v_j) \neq \gamma(v_i)\varphi(f, e_{ij})$. Suppose $\gamma(v_j) = (h, b)$ and $\gamma(v_i) = (g, a)$. Then $\gamma(v_i)\varphi(f, e_{ij}) = (g, a)\dot{f} = (gf, a)$. So the propriety condition on γ is that $(h, b) \neq (gf, a)$ for all $f \in \mathfrak{G}$. This can only be satisfied if $a \neq b$. We can therefore express the 0-free proper k-coloration $\gamma : V \to \mathfrak{G} \times E$ as $\gamma = (\gamma_{\mathfrak{G}}, \gamma_E)$ where γ_E is a proper k-coloration of Γ and $\gamma_{\mathfrak{G}} : V \to \mathfrak{G}$ is arbitrary. The number of proper k-colorations Γ_E is $\chi_\Gamma(k)$, so the number of 0-free proper k-colorations of $\mathfrak{G}\Gamma$ is $m^n\chi_\Gamma(k)$. Since $\lambda = mk$, we deduce that $\chi^{\mathrm{b}}_{\mathfrak{G}\Gamma}(\lambda) = m^n\chi_\Gamma(\lambda/m)$. This proves the first part of the theorem.

For the second part we put in the half edges to make the graph full. Then the color 0 is excluded, so $\chi_{\mathfrak{G}\Gamma^\bullet}(\lambda) = \chi^{\mathrm{b}}_{\mathfrak{G}\Gamma}(\lambda - 1)$. But $\lambda = km + 1$. So $k = \frac{\lambda - 1}{m}$, giving the result. □

Definition 64. The *Whitney numbers of the first kind* of a hyperplane arrangement, gain or biased graph, or matroid are the coefficients of the characteristic or chromatic polynomial, indexed in descending order. For instance,

$$\chi_\Phi(\lambda) = w_0(\Phi)\lambda^n + w_1(\Phi)\lambda^{n-1} + \cdots + w_{n-1}(\Phi)\lambda + w_n(\Phi).$$

The *balanced Whitney numbers of the first kind* of a gain or biased graph are the coefficients of the balanced chromatic polynomial, i.e.,

$$\chi_\Phi^{\mathrm{b}}(\lambda) = w_0^{\mathrm{b}}(\Phi)\lambda^n + w_1^{\mathrm{b}}(\Phi)\lambda^{n-1} + \cdots + w_{n-1}^{\mathrm{b}}(\Phi)\lambda + w_n^{\mathrm{b}}(\Phi).$$

From matroid theory we know these numbers alternate in sign: $(-1)^k w_k,\ (-1)^k w_k^{\mathrm{b}} \geq 0$. (Proof: Exercise.) They have combinatorial interpretations in terms of complex hyperplane arrangements, which we don't have time for now.

Example 65. Consider $\mathfrak{G}K_n^\bullet$. Since $\chi_{K_n}(k) = (k)_n = k(k-1)\cdots(k-[n-1])$, we get the chromatic polynomial formula

$$\begin{aligned}\chi_{\mathfrak{G}K_n^\bullet}(\lambda) &= m^n\Big(\frac{\lambda-1}{m}\Big)_n \\ &= m^n\Big(\frac{\lambda-1}{m}\Big)\Big(\frac{\lambda-1}{m}-1\Big)\cdots\Big(\frac{\lambda-1}{m}-[n-1]\Big) \\ &= (\lambda-1)(\lambda-1-m)(\lambda-1-2m)\cdots(\lambda-1-[n-1]m).\end{aligned}$$

We can expand this chromatic polynomial in terms of Stirling numbers of the first kind, $s(n,k)$, since $(x)_n = \sum_i s(n,i)x^i$:

$$\begin{aligned}\chi_{\mathfrak{G}K_n^\bullet}(\lambda) &= \sum_i s(n,i)(\lambda-1)^i m^{n-i} = \sum_i s(n,i)m^{n-i}\sum_j \binom{i}{j}\lambda^j(-1)^{i-j} \\ &= \sum_j \lambda^j \sum_i s(n,i)m^{n-i}\binom{i}{j}(-1)^{i-j} \\ &= \sum_k \lambda^{n-k}\Big[\sum_i s(n,i)m^{n-i}\binom{i}{n-k}(-1)^{i+k-n}\Big].\end{aligned}$$

Thus, the Whitney numbers of $\mathfrak{G}K_n^\bullet$ of the first kind are

$$w_k(\mathfrak{G}K_n^\bullet) = (-1)^k \sum_i s(n,i)m^{n-i}\binom{i}{n-k}(-1)^{n-i}.$$

The signs of the Stirling numbers are well known: $\operatorname{sgn} s(n,i) = (-1)^{n-i}$. In final form, then,

$$w_k(\mathfrak{G}K_n^\bullet) = (-1)^k \sum_{i=n-k}^{n} |s(n,i)|m^{n-i}\binom{i}{n-k}. \tag{2}$$

We'll do a similar calculation for an arbitrary group expansion in Example 72.

Example 66. Now suppose $\mathfrak{G} \leq K^\times$—as, for example, the finite cyclic group of order m is the group of m-th roots of unity in $\mathbb{C}$, or the cyclic group of order $q-1$ is the multiplicative group of the finite field $\mathbb{F}_q$. Then

$$\mathscr{H}[\mathfrak{G}K_n^\bullet] = \{h_i : x_i = 0\} \cup \{h_{ij}^g : x_j = x_i g \mid g \in \mathfrak{G}\}.$$

By Theorem 42, the characteristic polynomial of the arrangement is the chromatic polynomial of $\mathfrak{G}K_n^\bullet$, so

$$p_{\mathscr{H}[\mathfrak{G}K_n^\bullet]}(\lambda) = (\lambda-1)(\lambda-1-m)(\lambda-1-2m)\cdots(\lambda-1-[n-1]m).$$

From this we get a formula for the number of regions of the arrangement:

$$\begin{aligned}(-1)^n p_{\mathscr{H}[\mathfrak{G}K_n^\bullet]}(-1) &= (-1)^n m^n (\frac{-1-1m)_n}{=}(-1)^n m^n (\frac{-2}{m})_n \\ &= (2)(2+m)\cdots(2+[n-1]m).\end{aligned}$$

Example 67. Let's apply the preceding examples to the smallest nontrivial group: $\mathfrak{G} = \{\pm 1\}$. Then we are considering the signed graph $\pm K_n^\bullet$ (short for $\{\pm 1\}K_n^\bullet$). We infer that

$$p_{\mathscr{H}[\pm K_n^\bullet]}(\lambda) = 2^n \Big(\frac{\lambda - 1}{2}\Big)_n$$

and the number of regions is

$$(-1)^n p_{\mathscr{H}[\pm K_n^\bullet]}(-1) = (-1)^n 2^n \Big(\frac{-2}{2}\Big)_n = 2^n n!.$$

This has been long known to Lie theorists (who call regions "chambers"), but we have used a different and more general method to get this number.

The connection with Lie theory is historically important, as it was the impetus (stimulated by two questions from Richard Stanley) for the entire theory of gain-graphic matroids and hyperplane arrangements. A *root system* is a finite set of vectors in $\mathbb{R}^n$ that have certain nice integrality properties that I will not state here; they are stated in most books on Lie theory. The indecomposable root systems have been classified; they come in four infinite families, one for each dimension, called the *classical root systems*, and a small number of exceptional root systems. Our interest is in the classical root systems.

Example 68. Let's take our gain group to be $\{\pm 1\} \leq \mathbb{R}^\times$ and let's express the standard basis of $\mathbb{R}^n$ as $b_1, \ldots, b_n$. We associate vectors to the edges of our graph and we associate those vectors to their dual hyperplanes, i.e., the hyperplanes for which they are defining vectors. For an edge e_{ij} we write e_{ij}^+ if it is positive and e_{ij}^- if it is negative. For a half edge we write e_i (as it has no sign). We associate e_{ij}^+ to a vector $\pm(b_j - b_i)$, e_{ij}^- to a vector $\pm(b_j + b_i)$, and e_i to a vector $\pm b_i$. These vectors and their negatives constitute the root system B_n. The vectors $\pm(b_j - b_i)$ determine the hyperplane $x_i = x_j$, the vectors $\pm(b_j + b_i)$ determine the hyperplane $x_i = -x_j$, and the vectors $\pm b_i$ determine the hyperplane $x_i = 0$. These hyperplanes form the root system hyperplane arrangement $\mathscr{B}_n$. If we replace the half edges e_i by negative loops e_{ii}^-, we get vectors $\pm 2b_i$; this results in the root system C_n with hyperplane arrangement $\mathscr{C}_n = \mathscr{B}_n$. If we take only the positive edges, we get the root system called A_{n-1} (because it is not full-dimensional in $\mathbb{R}^n$) and its hyperplane arrangement $\mathscr{A}_{n-1}$. If we take only positive and negative links (i.e., no half edges or loops) we get the root system D_n and the arrangement $\mathscr{D}_n$. These correspond to the four infinite families of root systems.

The root system arrangements are signed-graphic. We list them with their graphs and the number of regions, r, computed from the chromatic polynomials. The circle in $\pm K_n^\circ$ means we add a negative loop instead of a half edge to every vertex (which makes a difference in computing vertex degrees—which we don't do in these notes).

$$\begin{aligned}\mathscr{A}_{n-1} = \mathscr{H}[K_n] &: r = (-1)^n \chi_{K_n}(-1) = (-1)^n (-1)_n = n!, \\ \mathscr{B}_n = \mathscr{H}[\pm K_n^\bullet],\ \mathscr{C}_n = \mathscr{H}[\pm K_n^\circ] &: r = (-1)^n \chi_{\pm K_n^\bullet}(-1) = 2^n n!, \\ \mathscr{D}_n = \mathscr{H}[\pm K_n] &: r = (-1)^n \chi_{\pm K_n}(-1) = 2^{n-1} n!.\end{aligned}$$

The last of these needs proof! The tool is the next theorem, which although simple is very convenient for computing chromatic polynomials.

Theorem 69 (Balanced Expansion)**.** *Let Φ be a gain graph without loose edges. Then*

$$\chi_\Phi(\lambda) = \sum_{W\subseteq V:\ W stable} \chi^{\mathrm{b}}_{\Phi:W^c}(\lambda-1).$$

The same holds for a biased graph Ω without loose edges.

In particular, $\chi_{\Phi^\bullet}(\lambda) = \chi^{\mathrm{b}}_\Phi(\lambda-1)$.

Proof. We leave the proof for a gain graph with finite gain group, and the general proof, as an exercise for the reader. □

Example 70 (Example 68 concluded.)**.** With Theorem 69 we can observe that

$$\chi_{\pm K_n}(\lambda) = \sum_{W\subseteq V:\ W \text{ stable}} \chi^{\mathrm{b}}_{\pm K_n:W^c}(\lambda-1).$$

But $W\subseteq V(\pm K_n)$ is stable only when (and when) $W=\emptyset$ or $|W|=1$, since the gain graph is compete and there are no half-edges or loops. So

$$\begin{aligned}
\chi_{\pm K_n}(\lambda) &= \sum_{\substack{W\subseteq V\\ W \text{ stable}}} \chi^{\mathrm{b}}_{\pm K_n:W^c}(\lambda-1)\\
&= \chi^{\mathrm{b}}_{\pm K_n}(\lambda-1) + n\chi^{\mathrm{b}}_{\pm K_n}(\lambda-1)\\
&= 2^n\chi_{K_n}(\lambda-1) + n2^{n-1}\chi_{K_{n-1}}(\lambda-1)\\
&= (\lambda-1)(\lambda-3)\cdots(\lambda-2n+1) + n(\lambda-1)(\lambda-3)\cdots(\lambda-2n+3)\\
&= (\lambda-1)(\lambda-3)\cdots(\lambda-2n+3)\cdot(\lambda+1-n).
\end{aligned}$$

Thus,

$$(-1)^n\chi_{\pm K_n}(-1) = (2)(4)\cdots(2(n-1))(n) = 2^{n-1}n!.$$

That gives the chromatic polynomial and the region count we wanted.

Example 71. By a similar computation (Exercise!), for any group $\mathfrak{G}$ of finite order m,

$$\chi^{\mathrm{b}}_{\mathfrak{G}K_n}(\lambda) = m^n\Big(\frac{\lambda}{m}\Big)_n$$

and

$$\chi_{\mathfrak{G}K_n}(\lambda) = m^{n-1}\Big(\frac{\lambda-1}{m}\Big)_{n-1}[\lambda-(m-1)(n-1)].$$

The Whitney numbers $w_k(\mathfrak{G}K_n)$ and $w^{\mathrm{b}}_k(\mathfrak{G}K_n)$ are an exercise for the reader.

Example 72. For later use we generalize Examples 65 and 71 to an arbitrary simple graph Γ, whose chromatic polynomial we write in the form $\chi_\Gamma(\lambda) = \sum_{i=0}^n w_i(\Gamma)\lambda^{n-i}$. All we need do is to replace the Stirling numbers by the Whitney numbers of Γ. Thus,

$$\begin{aligned}
w_k(\mathfrak{G}\Gamma^\bullet) &= (-1)^k\sum_i |w_i(\Gamma)|m^{n-i}\binom{i}{n-k},\\
w^{\mathrm{b}}_k(\mathfrak{G}\Gamma) &= m^{n-k}w_k(\Gamma);
\end{aligned}$$

and $w_k(\mathfrak{G}\Gamma)$ we leave as an exercise.

Exercise 73 (Supersolvable Group Expansions)**.** Let Γ be a simple graph and $\mathfrak{G}$ any group.

(a) Suppose Γ is chordal, i.e., $\operatorname{Lat}\Gamma$ is supersolvable. Is $\operatorname{Lat}\mathfrak{G}\Gamma^\bullet$ supersolvable? Is $\operatorname{Lat}\mathfrak{G}\Gamma$ supersolvable?

(b) Suppose Γ is not chordal. Is it possible for $\operatorname{Lat}\mathfrak{G}\Gamma^\bullet$ to be supersolvable? $\operatorname{Lat}\mathfrak{G}\Gamma$?

You may have to first decide what a modular coatom is in the gain graph.

As a bonus bit you could use supersolvability to get a nice formula for the chromatic polynomial.

Lecture 11: Affinographic Hyperplane Arrangements

4 December 2019
Notetaker: Michael Gottstein

For the gain-graphic arrangements we encountered previously, the gain group was multiplicative: $\mathfrak{G} \leq K^\times$ and the hyperplanes were homogeneous, i.e., subspaces of the vector space K^n. Now we switch to an additive group, $\mathfrak{G} \leq K^+$ to examine a new kind of gain-graphic arrangement, which I call "affinographic" because its hyperplanes are affine translates of graphic hyperplanes. This gives an affine—usually inhomogeneous—arrangement in the affine space $\mathbb{A}^n(K)$. For this type of arrangement we do not use half edges or loose edges.

Definition 74. A hyperplane of the form $x_j = x_i + c$ is called *affinographic*. An *affinographic hyperplane arrangement* is an arrangement whose hyperplanes are affinographic.

Definition 75. Given a gain graph Φ with gain group $\mathfrak{G} \leq K^+$, without half or loose edges, the corresponding affinographic hyperplane arrangement is

$$\mathscr{A}[\Phi] = \{a(e) : e \in E\},$$

where a is a function that gives a hyperplane

$$a(e_{ij}) : x_j - x_i = \varphi(e_{ij}),$$

or equivalently $x_j = x_i + \varphi(e_{ij})$, for each edge of Φ.

Recall from Lecture 7 (Definition 41) that $\operatorname{Lat}^{\mathrm{b}} \Phi = \{A \in \operatorname{Lat} \Phi : A \text{ is balanced}\}$.

Theorem 76. *Let $S \subseteq E$. Then $\bigcap a(S) \neq \varnothing$ if and only if S is balanced.*
The function a gives a semilattice isomorphism $\operatorname{Lat}^{\mathrm{b}} \Phi \cong \mathscr{L}(\mathscr{A}[\Phi])$.

Proof. We start the proof with three useful lemmas.

Lemma 77. *If C is an unbalanced circle, then $\bigcap a(C) = \varnothing$.*

Proof. Let $C = v_0 e_{01} v_1 e_{12} v_2 \dots e_{l-1,l} v_l$, where $v_0 = v_l$. Then $x \in \bigcap a(C) \iff x$ satisfies all the equations

$$\begin{aligned} x_1 &= x_0 + \varphi(e_{01}), \\ x_2 &= x_1 + \varphi(e_{12}), \\ &\dots \\ x_l &= x_{l-1} + \varphi(e_{l-1l}), \end{aligned} \tag{3}$$

hence

$$x_l = x_0 + \varphi(e_{01}) + \varphi(e_{12}) + \dots + \varphi(e_{l-1,l}) = x_0 + \varphi(C).$$

But $x_l := x_0$, so this is impossible if $\varphi(C) \neq 0$, i.e., when C is unbalanced. Thus $\bigcap a(C) = \varnothing$. □

Lemma 78. *If $S \subseteq E$ and $F \subseteq S$ is a maximal forest in S, then $\bigcap a(S) = \bigcap a(F)$.*

Proof. For a balanced circle, the equation $x_l = x_{l-1} + \varphi(e_{l-1,l})$ is implied by the others in Equation (3). Indeed, from the first $l-1$ of those equations we infer that $x_{l-1} = x_0 + \varphi(e_{01}e_{12}....e_{l-2,l-1}) = x_0 + \varphi(C) - \varphi(e_{l-1,l})$. Since C is balanced and since $x_l = x_0$, this quantity $= x_0 + 0 - \varphi(e_{l-1,l}) = x_l - \varphi(e_{l-1,l})$. Thus, $x_{l-1} = x_l - \varphi(e_{l-1,l})$, which is the desired equation.

This implies that if $x \in \bigcap a(C \backslash e_{l-1,l})$, then $x \in a(e_{l-1.l})$. That is, $\bigcap a(C \backslash e_{l-1,l}) \subseteq a(e_{l-1,l})$.

Now, for edge sets F and S as in the hypothesis, for each $e \in S \setminus F$ there is a circle $C \subseteq F \cup \{e\}$ that contains e. By the preceding calculation, $a(e) \supseteq \bigcap a(C \setminus e) \supseteq \bigcap a(F)$. It follows that $\bigcap_{e \in S \setminus F} a(e) \supseteq \bigcap a(F)$. So, $\bigcap a(S) \supseteq \bigcap a(F)$. As the reverse inclusion is obvious, we have equality. □

Lemma 79. *For a forest $F \subseteq E$, $\bigcap a(F)$ is an affine flat whose codimension is $\#F$.*

Proof. We induct on the number of edges in F.

If there are no edges then the codimension is obviously 0.

A forest with at least one edge has a vertex of degree 1, say v_k with edge e_{mk}. The hyperplane $a(e_{mk})$ is given by the equation $x_k = x_m + \varphi(e_{mk})$, and no other hyperplane in $a(F)$ has x_k in its equation. Consequently, x_k is unrestricted for $x \in \bigcap a(F \setminus e_{mk})$, from which we conclude that

$$a(e_{mk}) \not\supseteq \bigcap a(F \setminus e_{mk})$$

and in $\bigcap a(F)$ we are imposing only the new restriction $x_k = x_m + \varphi(e_{mk})$, from which it follows that

$$a(e_{mk}) \cap \bigcap a(F \setminus e_{mk}) \neq \varnothing.$$

Since $a(e_{mk}) \cap \bigcap a(F \setminus e_{mk}) \neq \varnothing$, the modular law of dimension in $\mathbb{A}^n(K)$ applies; therefore $\operatorname{codim} \bigcap a(F) = \operatorname{codim} \bigcap a(F \setminus e_{mk}) + 1$, so by induction we have the result. □

Now we prove the theorem.

Case 1: S is unbalanced. Then $S \supseteq C$, an unbalanced circle, and $\bigcap a(S) \subseteq \bigcap a(C) = \varnothing$ by Lemma 77.

Case 2: S is balanced. Let F be a maximal forest in S. Then $\bigcap a(S) = \bigcap a(F)$ by Lemma 78, which is not empty by Lemma 79. And $\operatorname{codim} \bigcap a(S) = \operatorname{codim} \bigcap a(F) = \#F$ by Lemma 79.

By elementary graph theory $\#F = n - c(F)$, which $= n - c(S)$ since F is maximal in S. Therefore $\operatorname{rk} \bigcap a(S)$, in $\mathscr{L}(\mathscr{A}[\Phi])$, is equal to $n - c(S) = \operatorname{rk}_\Phi(S)$ (in the frame matroid). Since the ranks match, the closure deduced from $\mathscr{A}[\Phi]$ for balanced edge sets is the same as that in $\mathbf{F}(\Phi)$ for balanced edge sets. This implies that the closed sets in $\mathbf{F}(\Phi)$ that are balanced are in one-to-one correspondence (via the mapping a) with the flats of $\mathscr{A}[\Phi]$. □

Lecture 12: The Lift Matroid, with Examples and Modular Coloring

6 December 2019
Notetaker: Andrew Lamoureux

Characteristic polynomial. Let's begin with one of the main theorems about affinographic arrangements—which (at last) gives a solid justificiation for the balanced chromatic polynomial.

Theorem 80. *For a field K, a group $\mathfrak{G} \leq K^+$, and a $\mathfrak{G}$-gain graph Φ, the characteristic polynomial of the affinographic arrangement of Φ is $p_{\mathscr{A}[\Phi]}(\lambda) = \chi_\Phi^{\mathrm{b}}(\lambda)$.*

Proof. For simplicity we write $\mathscr{A} := \mathscr{A}[\Phi]$. Recall that for a balanced subset $S \subseteq E := E(\mathfrak{G})$ and for $\mathscr{S}$ the corresponding set of hyperplanes in $\mathscr{A}$,

$$\dim \bigcap \mathscr{S} = n - \operatorname{rk} \bigcap \mathscr{S} = n - \operatorname{rk}(S) = b(S) = c(S).$$

Then from the polynomial definitions and the balance property in Theorem 76,

$$p_{\mathscr{A}}(\lambda) = \sum_{\mathscr{S} \subseteq \mathscr{A} : \bigcap \mathscr{S} \neq \emptyset} (-1)^{|\mathscr{S}|} \lambda^{\dim(\bigcap \mathscr{S})} = \sum_{S \subseteq E:\, S \text{ balanced}} (-1)^{|S|} \lambda^{b(S)} = \chi_\Phi^{\mathrm{b}}(\lambda). \qquad \square$$

The lift matroid. Now we put the projectivization of an affine arrangement to serious use. We will examine the projectivization of $\mathscr{A}[\Phi]$, written $\mathscr{A}[\Phi]_{\mathbb{P}}$ or $\mathscr{A}_{\mathbb{P}}[\Phi]$, and infer from it a new matroid of a gain graph.

Recall from Stanley's lectures that for an affine hyperplane (or subspace) h in $\mathbb{A}^n(K)$, $h_{\mathbb{P}}$ is its extension into the projective space $\mathbb{P}^n(K)$. For an affine hyperplane arrangment $\mathscr{A}$ in $\mathbb{A}^n(K)$, $\mathscr{A}_{\mathbb{P}} := \{h_{\mathbb{P}} \mid h \in \mathscr{A}\} \cup \{h_\infty\}$, a hyperplane arrangement in $\mathbb{P}^n(K)$.

Note that h_∞, the ideal hyperplane, is isomorphic to $\mathbb{P}^{n-1}(K)$. The arrangement *induced* in h_∞ by $\mathscr{A}_{\mathbb{P}}$ is $\mathscr{A}_{\mathbb{P}}^{h_\infty} := \{h_{\mathbb{P}} \cap h_\infty \mid h \in \mathscr{A}\}$; its matroid is denoted, as usual, by $\mathscr{M}(\mathscr{A}_{\mathbb{P}}^{h_\infty})$. It follows that $\mathscr{M}(\mathscr{A}_{\mathbb{P}}^{h_\infty}) \cong \mathscr{M}(\mathscr{A}_{\mathbb{P}})/h_\infty$, the contraction matroid, by the natural correspondence $h_{\mathbb{P}} \mapsto h_{\mathbb{P}} \cap h_\infty$. Recall also that $h_{\mathbb{P}} \cap h_\infty = h'_{\mathbb{P}} \cap h_\infty$ if and only if h and h' are parallel.

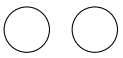
We are going to give an intrinsic characterization of the matroid $\mathbf{L}_0(\Phi)$ implied by the projectivization $\mathscr{A}_{\mathbb{P}}[\Phi]$. The first step is to state that characterization; then we prove it is naturally isomorphic to $\mathscr{M}(\mathscr{A}_{\mathbb{P}}[\Phi])$.

Let $E_0 := E \cup \{e_0\}$, where e_0 is a new object that is not in either E or V. We extend the notion of balance to E_0: we call $S \subset E_0$ *balanced* if $S \subseteq E$ and S is balanced as a subset of E; any other subset of E_0 is *unbalanced*.

Theorem 81. *For any gain graph Φ, there is a matroid $\mathbf{L}_0(\Phi)$ with ground set E_0 defined by any of the following five equivalent axioms. This definition also applies to any biased graph Ω.*

I. A set $S \subseteq E_0$ is a circuit *of $\mathbf{L}_0$ if and only if it is of one of the following types:*
  i. a balanced circle,
  ii. a contrabalanced tight handcuff (or tight bracelet),

iii. a contrabalanced loose bracelet,



iv. a contrabalanced theta graph;

v. $C \cup \{e_0\}$ *for an unbalanced circle* C.

II. *A set* $S \subseteq E_0$ *is an* independent set *of* $\mathbf{L}_0$ *if and only if it is of one of the following types:*

i. a forest,

ii. an unbalanced unicyclic graph,

iii. $F \cup \{e_0\}$ *for any forest* F.

III. *The* rank function *of* $\mathbf{L}_0$ *is*

$$\mathrm{rk}_{\mathbf{L}}(S) = \begin{cases} n - c(S), & \text{if } S \text{ is balanced,} \\ n - c(S) + 1, & \text{otherwise.} \end{cases}$$

IV. *The* lattice of flats *is* $\operatorname{Lat} \mathbf{L}_0(\Phi) = \operatorname{Lat}^{\mathrm{b}}(\Phi) \cup \operatorname{Lat}_0(\Gamma)$, *where* $\Gamma = \|\Phi\|$, *the underlying graph of* Φ, *and* $\mathbf{L}_0(\Gamma) := \{A \cup \{e_0\} | A \in \operatorname{Lat}(\Gamma)\}$.

V. *The* closure *of* $S \subseteq E_0$ *is*

$$\mathrm{cl}(S) = \begin{cases} \mathrm{bcl}(S), & \text{if } S \text{ is balanced,} \\ \mathrm{cl}_\Gamma(S) \cup \{e_0\}, & \text{if } S \subseteq E \text{ is unbalanced,} \\ \mathrm{cl}_\Gamma(S \setminus \{e_0\}) \cup \{e_0\}, & \text{if } e_0 \in S. \end{cases}$$

Not a Proof. Sadly, we will not prove this theorem; the proof is too long. We will, however, prove in Theorem 83 that $\mathbf{L}_0$ is the right matroid for $\mathscr{A}_{\mathbb{P}}[\Phi]$, which incidentally proves it is a matroid when Φ has gain group K^+. □

Definition 82 (Lift matroid)**.** The *extended lift matroid* of Φ is the matroid $\mathbf{L}_0(\Phi)$ defined in Theorem 81. The *lift matroid* $\mathbf{L}(\Phi)$ is the restriction of $\mathbf{L}_0(\Phi)$ to the ground set E.

It should be clear that if Φ (or Ω) is balanced, the lift matroid $\mathbf{L}$ is the same as the ordinary graphic matroid and as the frame matroid. It is more difficult to say when the lift matroid coincides with the frame matroid. The obvious necessary and sufficient condition is that there be no two vertex-disjoint unbalanced circles in any component of the graph; but characterizing the gain graphs or biased graphs that meet that condition is hard and presently unsolved (except for signed graphs–gain group $\{\pm 1\}$–for which see [7].

Now, define $a_{\mathbb{P}}$ to be the projective extension to E_0 of the function $a : E \to \mathscr{A}[\Phi]$ defined in Lecture 11. This function satisfies

$$a_{\mathbb{P}}(e) = \begin{cases} h_{\mathbb{P}}, & \text{when } h = a(e) \text{ for some } e \in E, \\ h_\infty, & \text{when } e = e_0. \end{cases}$$

That is, $a_{\mathbb{P}}(e) = a(e)_{\mathbb{P}}$ for an edge e of Φ.

Theorem 83. *The mapping* $a_{\mathbb{P}}$ *is an isomorphism* $\mathbf{L}_0(\Phi) \to \mathscr{M}(\mathscr{A}_{\mathbb{P}}[\Phi])$.

We give two proofs that share steps but rely on different characterizations of the matroids.

First Proof. For simpler notation we write $\mathscr{A}$ for $\mathscr{A}[\Phi]$, $\mathbf{L}_0 = \mathbf{L}_0(\Phi)$, and $\mathbf{F} = \mathbf{F}(\Phi)$ (the frame matroid). We show $\mathscr{M}(\mathscr{A}_{\mathbb{P}}) \cong_{a_{\mathbb{P}}} \mathbf{L}_0(\Phi)$ by showing the ranks in the two matroids are

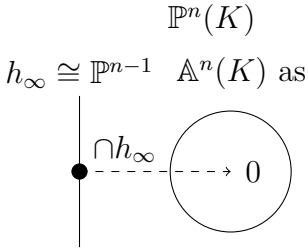


FIGURE 12. The mapping $\cap h_\infty$ inverts to give a unique vector subspace of K^n.

the same. Let $S \subseteq E_0$ be balanced; then

$$\begin{aligned}\operatorname{rk}_M \bigcap a_{\mathbb{P}}(S) &= \operatorname{codim} \bigcap a_{\mathbb{P}}(S) = \operatorname{codim} \bigcap a(S) = \operatorname{rk} \bigcap a(S)\\ &= \operatorname{rk}_{\mathbf{F}(\Phi)} S = n - b(S) = n - c(S) = \operatorname{rk}_{\mathbf{L}}(S),\end{aligned}$$

where the second equality follows from $\bigcap a(S) \neq \emptyset$ and the fourth from the second part of Theorem 76.

Suppose S is unbalanced, but $e_0 \notin S$. The graphic hyperplane $h(e_{ij}) : x_i = x_j$ in $\mathbb{A}^n(K)$ has projective extension $h_{\mathbb{P}}(e_{ij})$ in $\mathbb{P}^n(K)$ that satisfies $\{x_j = x_i\}_{\mathbb{P}} \cap h_\infty = \{x_j = x_i + c\}_{\mathbb{P}} \cap h_\infty$ for all $c \in K$ because the hyperplanes are parallel. Hence, $a_{\mathbb{P}}(e) \cap h_\infty = h_{\mathbb{P}}(e) \cap h_\infty$ for any edge $e \in E$. This, in particular, implies that $\mathscr{A}_{\mathbb{P}}^{h_\infty} = \mathscr{H}[\Gamma]_{\mathbb{P}}^{h_\infty}$. Because $\mathscr{H}[\Gamma]$ is homogeneous, the mapping $\mathscr{H}[\Gamma] \xrightarrow{\cap h_\infty} \mathscr{H}_{\mathbb{P}}[\Gamma]^{h_\infty}$ is a rank-preserving bijection. That bijection implies a matroid isomorphism $\mathscr{M}(\mathscr{H}[\Gamma])_{\mathbb{P}}^{h_\infty} \cong \mathscr{M}(\mathscr{H}[\Gamma])$ (see Figure 12).

Now consider $S = T \cup \{e_0\}$ for some $T \subseteq E$. Then $\bigcap a_{\mathbb{P}}(S) \subseteq a_{\mathbb{P}}(e_0) = h_\infty$, so

$$\bigcap a_{\mathbb{P}}(S) = \bigcap_{e \in T} \big(a_{\mathbb{P}}(e) \cap h_\infty\big) = \bigcap_{e \in T} \big(h_{\mathbb{P}}(e) \cap h_\infty\big) = h_\infty \cap \bigcap_{e \in T} h_{\mathbb{P}}(e_{ij}).$$

Thus the rank of $\bigcap a_{\mathbb{P}}(S)$ is

$$\operatorname{rk} \bigcap a_{\mathbb{P}}(S) = \operatorname{codim} \bigcap a_{\mathbb{P}}(S) = \underbrace{\operatorname{codim} \bigcap h_{\mathbb{P}}(S)}_{n - c(T)} + \underbrace{1}_{h_\infty}. \qquad \square$$

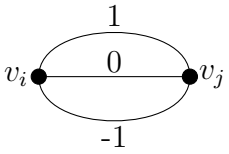
Second Proof. Again we write $\mathscr{A}$ for $\mathscr{A}[\Phi]$. This proof depends on showing that the closed sets of $\mathbf{L}_0(\Phi)$ are the right ones for $\mathscr{M}(\mathscr{A}_{\mathbb{P}})$. For balanced closed sets, this is Theorem 76.

For unbalanced ones, since they all contain e_0, which corresponds to h_∞, they must correspond to ideal flats of $\mathscr{M}(\mathscr{A}_{\mathbb{P}})$; in other words, subspaces in $\mathscr{L}(\mathscr{A}_{\mathbb{P}}^{h_\infty})$. Such a flat is the intersection with h_∞ of a set of hyperplanes $h_{\mathbb{P}}$ for $h \in \mathscr{S}$ where $\mathscr{S}$ is some subset of $\mathscr{A}$. The affine hyperplane h has equation $x_j - x_i = c$ for a constant c, but its ideal part, $h_{\mathbb{P}} \cap h_\infty$, is independent of c; so we may replace $\mathscr{A}[\Phi]$ by the arrangement $\mathscr{H}[\Gamma]$ of graphic hyperplanes $h(e) : x_j = x_i$ that are parallel to the hyperplanes $a(e)$ of $\mathscr{A}$. Since $\mathscr{H}[\Gamma]$ is homogeneous, the flats $s \in \mathscr{L}(\mathscr{H}_{\mathbb{P}}[\Gamma])$ are determined by their ideal parts $s_\infty := s_{\mathbb{P}} \cap h_\infty$. Therefore, $\mathscr{L}(\mathscr{A}_{\mathbb{P}}^{h_\infty}) \cong \mathscr{L}(\mathscr{H}_{\mathbb{P}}[\Gamma]^{h_\infty}) \cong \mathscr{L}(\mathscr{H}[\Gamma]) \cong \operatorname{Lat}\Gamma \cong \operatorname{Lat}_0\Gamma$ with the third isomorphism given by $a_{\mathbb{P}}$. That proves the ideal flats of $\mathscr{A}_{\mathbb{P}}$ correspond to the unbalanced flats of $\mathbf{L}_0(\Phi)$ via $a_{\mathbb{P}}$. That completes the proof. $\square$

Popular Affinographics. Several affinographic arrangements that have received a lot of attention in recent years are the real affine arrangements of certain *integral gain graphs*—where the gain group is the additive group of integers, $\mathbb{Z}^+$, regarded as a subgroup of $\mathbb{R}^+$. I will describe some of them. In each example I state the gain graph Φ; the arrangement is $\mathscr{A}[\Phi]$.

These graphs are a kind of partial group expansion: expansions of a base graph Δ by subsets of the gain group. To state the gains we assume a special orientation of the base graph: the vertex set is $V = \{v_1, \ldots, v_n\}$ and edges are oriented upwards, i.e., from v_i to v_j where $i < j$; we denote this oriented graph by $\vec{\Delta}$. (Actually, we always use $\vec{K}_n$.) Then, for instance, the notation $\{1, 2, -3\}\vec{\Delta}$ means that each edge e is replaced by three edges with gains 1, 2, and -3 in the upward direction; equivalently, the gains are -1, -2, $+3$ in the downward direction.

Example 84. The *Catalan arrangement* is associated with the *Catalan gain graph* $\{0, \pm 1\}\vec{K}_n$, or $\{0, \pm 1\}K_n$. (The arrow over K is superfluous because the gain set is symmetric.) The picture of an expanded edge e_{ij} (with $i < j$) is

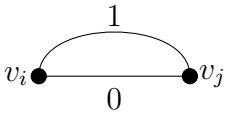


This arrangement gets its name from the curious fact that the number of regions is a Catalan number.

A variation is the *hollow Catalan arrangement*, with gain graph $\{\pm 1\}\vec{K}_n$. The picture is the same except that the edge with gain 0 is missing.

A more elaborate variant is the *extended Catalan arrangement*, whose gain graph is $\Phi = \{0, \pm 1, \ldots, \pm l\}\vec{K}_n$. It has a hollow version as well, without the 0-edges.

Example 85. The *Shi arrangment* has the gain graph $\{0, +1\}\vec{K}_n$. A picture is



The absence of sign symmetry in the gains (i.e., the fact that there is a $+1$ edge e_{ij} but no -1 edge e_{ij}) makes it more difficult to compute the Shi characteristic polynomial than the Catalan arrangement's.

Example 86. The *Linial arrangement*[2] accompanies $\{+1\}\vec{K}_n$.

The Shi and Linial arrangements also have extended variants, though their definitions are not obvious.

Modular coloring. We wish to compute the characteristic polynomial $p_{\mathscr{A}}(\lambda)$ of each arrangement in our list, but there is a difficulty: we cannot count proper colorations in an infinite group like $\mathbb{Z}^+$. The solution is to compute $\chi^{\mathrm{b}}_{\Phi}(\lambda)$ using colors in $\mathbb{Z}^+_m$, which is the

[2]Named for Nathan Linial.

additive group of integers modulo m, using the next proposition. For a $\mathbb{Z}^+$-gain graph Φ, define Φ/m to have the same underlying graph and gains modulo m; that is, $\varphi_{\Phi/m}(e) := \varphi_\Phi(e) \bmod m$. These are *modular gains*.

Theorem 87. *For a $\mathbb{Z}^+$-gain graph Φ, $\chi^{\mathrm{b}}_\Phi(\lambda) = \chi^{\mathrm{b}}_{\Phi/m}(\lambda)$ if, and only if, m does not divide the gain of any unbalanced circle in Φ.*

Proof. Let $\langle\Phi\rangle$ be the biased graph of Φ. Then $\chi^{\mathrm{b}}_\Phi(\lambda) = \chi^{\mathrm{b}}_{\langle\Phi\rangle}(\lambda)$ and $\chi^{\mathrm{b}}_{\Phi/m}(\lambda) = \chi^{\mathrm{b}}_{\langle\Phi/m\rangle}(\lambda)$ by the definition of χ^{b}. Also, $\langle\Phi\rangle = \langle\Phi/m\rangle$ if and only if m is not a divisor of the gain of any unbalanced circle, as then unbalanced circles are unchanged by passing from Φ to Φ/m. This implies sufficiency.

For necessity, consider the class $\mathscr{S}$ of unbalanced edge sets that become balanced modulo m. Then

$$\chi^{\mathrm{b}}_{\Phi/m}(\lambda) - \chi^{\mathrm{b}}_\Phi(\lambda) = \sum_{S\in\mathscr{S}} (-1)^{|S|}[\lambda^{b_{\Phi/m}(S)} - \lambda^{b_\Phi(S)}] = \sum_{S\in\mathscr{S}} (-1)^{|S|}\lambda^{b_\Phi(S)}[\lambda^{b_{\Phi/m}(S)-b_\Phi(S)} - 1] \tag{4}$$

Let $b_0 := \min\{b(S) : S \in \mathscr{S}\}$ and $\mathscr{S}_0 := \{S \in \mathscr{S} : b(S) = b_0\}$. Then the term of degree b_0 in Equation (4) has coefficient $-|\mathscr{S}_0|$; that is, the coefficient of λ^{b_0} is reduced by $|\mathscr{S}_0|$ in passing from Φ to Φ/m. This proves that equality fails if any circle becomes balanced upon going to modular gains. □

The modular strategy for computing the balanced chromatic polynomial is to find infinitely many "good" values m, not dividing any circle gain, at which to calculate $\chi^{\mathrm{b}}_\Phi(m)$ by group coloring using the finite cyclic group $\mathbb{Z}^+_m$. We obtain $\chi^{\mathrm{b}}_{\Phi/m}(m)$ by counting proper $\mathbb{Z}^+_m$-colorations, and when m is a good modulus this number equals $\chi^{\mathrm{b}}_\Phi(m)$. Doing this for n good moduli m determines the balanced chromatic polynomial, as we know the degree (n) and the leading coefficient (1). (In practice the same counting procedure succeeds for all $m > \max_{C\in\mathscr{C}(\Phi)} \varphi(C)$ so there is no advantage to restricting to only n moduli. For the same reason there is no advantage to k-coloring; 1-coloring is simpler and sufficient.)

Exercise 88 (The Catalan Polynomial)**.** Compute the characteristic polynomial of the Catalan arrangement by using modular group coloring of the associated gain graph.

Exercise 89 (The Shi Polynomial)**.** Compute the characteristic polynomial of the Shi arrangement by using modular group coloring of the associated gain graph. The Catalan computation might be helpful, depending on what approach you take.

This was a significant research question for a time.

Exercise 90 (The Hollow Catalan Polynomial)**.** Compute the characteristic polynomial of the hollow Catalan arrangement.

Lecture 13: Modular Coloring for the Catalan Arrangement

22 January 2020
Notetaker: Nicholas Lacasse

Review. We consider affine hyperplane arrangements in $\mathbb{A}^n(\mathbb{R})$, in particular those that arise from an integral gain graph, that is, Φ with the additive gain group $\mathbb{Z}^+$. An edge $e = v_iv_j$ with gain g in Φ gives the hyperplane $x_j - x_i = g$. (Note that this makes the [arbitrary] orientation of the edge significant. If the edge were oriented in the opposite direction with gain g, it would give the hyperplane $x_i - x_j = g$.) The hyperplane arrangement determined by Φ is written $\mathscr{A}[\Phi]$.

The Catalan arrangement $\mathscr{C}_n$ is $\{x_j - x_i = 0, 1, -1 \text{ for } i < j\}$ in $\mathbb{A}^n(\mathbb{R})$ where $\mathbb{A}^n$ denotes n dimensional affine space. The gain graph corresponding to the Catalan arrangement is $\{0, \pm 1\}\vec{K}_n$, that is, K_n with three edges, bearing gains 0, 1, and -1, between each pair of vertices. We call it the *Catalan gain graph*. (Here $\vec{K}_n$ denotes K_n with vertex set $\{v_1, v_2, \ldots, v_n\}$ and all edges oriented upward for the assignment of gains. The same convention can be applied to any graph.) To show why this notation is useful, I mention the Shi arrangement, $\mathscr{S}_n := \mathscr{A}[\{0, 1\}\vec{K}_n]$, whose hyperplanes are $x_j - x_i = 0, 1$ for $i < j$.

Our goal is to compute the characteristic polynomial of the Catalan and related arrangements. We achieve this by using three previous theorems. The first two theorems are:

Theorem 91 (Theorem 80)**.**

$$p_{\mathscr{A}[\Phi]}(\lambda) = \chi^{\mathrm{b}}_{\Phi}(\lambda) := \sum_{S \subseteq E:\ S \text{ balanced}} (-1)^{\#S} \lambda^{b(S)}.$$

Theorem 92 (Theorem 44 with $k = 1$)**.** *If $\mathfrak{G}$ is a finite group, then $\chi^{\mathrm{b}}_{\Phi}(\#\mathfrak{G})$ is the number of proper $\mathfrak{G}$-colorations of Φ.*

A proper $\mathfrak{G}$-coloration of Φ is a mapping $\gamma : V(\Phi) \to \mathfrak{G}$ such that, for each edge $e = v_iv_j$, say with gain g, then $\gamma(v_j) \neq \gamma(v_i)g$. It is a proper 1-coloration with simpler notation.

These results let us use coloring methods to determine the characteristic polynomial of an arrangement. However, if our group is, like $\mathbb{Z}^+$, infinite, then so is the number of proper colorations. That creates an obvious difficulty. Fortunately, we have a third theorem to deal with the difficulty. Let $\mathscr{B}(\Phi)$ denote the set of balanced circles of Φ.

Theorem 93 (Definition 41)**.** *Suppose the underlying graphs of Φ and Φ' are the same and moreover $\mathscr{B}(\Phi) = \mathscr{B}(\Phi')$. Then $\chi^{\mathrm{b}}_{\Phi}(\lambda) = \chi^{\mathrm{b}}_{\Phi'}(\lambda)$.*

So if we can change the gains on the Catalan gain graph so they are in a finite group, without changing the list of balanced circles, then we may get a meaningful count of proper colorations. This can be done. The idea is to take the integral gains modulo m for $m > n$, changing the gain group from $\mathbb{Z}^+$ to $\mathbb{Z}^+_m$. This will not destroy balance of any circle because if a circle has gain 0 in $\mathbb{Z}$, it has gain 0 modulo m. It will not create new balanced circles because, since the largest magnitude of a gain is 1, no circle has gain larger than n. We formulate this method as a lemma.

Theorem 94. *Suppose Φ is an integral gain graph and $m \in \mathbb{Z}_{>0}$ is not the gain of any circle in Φ. Let Φ/m be the same gain graph with gains interpreted modulo m, so the gain group is $\mathbb{Z}_m$. Then $\chi^{\mathrm{b}}_{\Phi}(\lambda) = \chi^{\mathrm{b}}_{\Phi/m}(\lambda)$ and $\chi_{\Phi}(\lambda) = \chi_{\Phi/m}(\lambda)$.*

Catalan calculations. We are now tasked with counting the number of proper $\mathbb{Z}_m^+$-colorations of $\{0, \pm 1\}K_n$. This means we need to count functions $\gamma : V = \{v_1, \ldots, v_n\} \to \mathbb{Z}_m^+$ such that $\gamma(v_j) - \gamma(v_i) \neq 0, \pm 1$ for all $i \neq j$. We encourage the reader to close the "book" and attempt to work out a solution before continuing.

Here is our class's solution to the coloring problem. We view the vertices v_i as objects that we will be placing into bins. The bins are labeled with integers from 0 to $m-1$. No two vertices may be placed in the same bin, so there will be $m-n$ empty bins. Let us label the empty bins with the integers from 0 to $m-n-1$. Now fix vertex v_1 in the space to the left of bin 0 and we place the remaining vertices in the spaces between empty bins, at most one to each space. There are $m-n-1$ such spaces and we choose $n-1$ of them for vertices, in $\binom{m-n}{n-1}$ possible ways. Those vertices may be permuted in any order, giving us a factor of $(n-1)!$. Now we have a sequence of length m that consists of n vertices and $m-n$ empty bins, with v_1 in position 0. Assign each vertex the number that is its position in this sequence; thus each v_i gets a label in $\mathbb{Z}_m$. To allow for the m ways v_1 could be labelled, we can shift the whole pattern cyclically by any amount from 0 to $m-1$. This gives a total number of labellings equal to $m\binom{m-n-1}{n-1}(n-1)!$. Each labelling is a proper $\mathbb{Z}_m^+$-coloration of $\{0, \pm 1\}K_n/m$ and we obtain every such proper coloration.

Let C_n denote the n-th Catalan number: $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proposition 95. *For the Catalan arrangement $\mathscr{C}_n$:*
(1) $p_{\mathscr{C}_n}(\lambda) = \lambda(\lambda - n - 1)_{n-1}$.
(2) *$\mathscr{C}_n$ has $n!C_n$ regions.*

Proof. By Theorem 92 we have found the balanced chromatic polynomial of $\{0, \pm 1\}K_n/m$. The first conclusion follows by Theorems 87, 93, and 91.

The second part follows, according to [9, Theorem 2.5], by calculating $(-1)^n p_{\mathscr{C}_n}(-1)$. □

Exercise 96. How many bounded regions does the Catalan arrangement have?

The factor $n!$ in $r(\mathscr{C}_n) = n!C_n$ is the number of regions of the complete-graph arrangement $\mathscr{A}[0K_n])$. Since the latter is a subarrangement of the Catalan arrangement, each region of the Catalan arrangement is a piece of a region of the complete-graph arrangement. The numerical relationship $r(\mathscr{C}_n) = C_n r(\mathscr{A}[0K_n]))$ makes us wonder if there is a combinatorial relationship. Lo and behold!

Corollary 97. *Each region of the complete-graph arrangement $\mathscr{A}[\{0\}K_n]$ contains the same number of regions of the Catalan arrangement and also the same number of bounded regions.*

Thus, the Catalan arrangement divides a region of the complete-graph arrangement into C_n subregions. This is the source of its name.

Proof. One can see from the defining equations of the hyperplanes and the defining inequalities of the regions that the symmetric group $\mathfrak{S}_n$ acts freely and transitively on the regions of the complete-graph arrangement and also on those of the Catalan arrangement.

In more detail: The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{A}^n(R)$ by permutation of the coordinates and this action preserves both the complete-graph arrangement and the Catalan arrangement. Furthermore, since each region of $\mathscr{A}[0K_n]|$ consists of the points with a certain linear ordering of coordinates, $x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}$ for some $\sigma \in \mathfrak{S}_n$, the action preserves whole regions and is transitive on the regions (indeed, sharply transitive). The consequence

is that every region of $\mathscr{A}[0K_n]$ is dissected by $\mathscr{C}_n$ in the same way, so into the same number of Catalan regions, and the same number of bounded Catalan regions. □

Graphic arrangements. The complete-graph arrangement has two formulas: $\mathscr{A}[\{0\}K_n]$ and $\mathscr{H}[\{1\}K_n]$; these are the same arrangement. It is, respectively, both an affinographic arrangement (from a lift matroid) and a homogeneous arrangement (from a frame matroid). We call this the *complete-graph arrangement* or K_n*-arrangement* for short.[3] In each case we have K_n with all gains equal to the neutral element, so the frame and lift matroids are identical. The same equation holds for any graph, not only K_n. We call these arrangements *graphic*. They have been much studied; in particular they are well defined over every field of every characteristic.

Related to Catalan. Here is a closely related arrangement, with a nice exercise.

Example 98. The *hollow Catalan arrangement* is $\mathscr{C}_n^\circ = \mathscr{A}[\{\pm 1\}K_n]$. That is, it is the Catalan arrangement without the graphic hyperplanes $x_i = x_j$.

Exercise 99. Calculate the characteristic polynomial $p_{\mathscr{C}_n^\circ}(\lambda)$ and the number of regions of $\mathscr{C}_n^\circ$. How many bounded regions does the hollow Catalan arrangement have?

And here is another related arrangement with (naturally) another exercise.

Example 100. The *extended Catalan arrangement* for a positive integer k is

$$\mathscr{C}_{n,k} = \mathscr{A}[\{0, \pm 1, \pm 2, \ldots, \pm k\}K_n].$$

There is also the hollow extended Catalan arrangement, $\mathscr{C}_{n,k}^\circ$, whose definition is obvious.

Exercise 101. Calculate the characteristic polynomial and the number of regions of $\mathscr{C}_{n,k}$. How many bounded regions are there?

[3]It is widely known as the "braid arrangement" but that name really belongs only to the complex K_n-arrangement.

Lecture 14: Finite Field Method vs. Modular Coloring

24 January 2020
Notetaker: Mike Gottstein

Example: The hollow Catalan arrangement. The hollow Catalan arrangement is $\mathscr{A}[\{\pm 1\}K_n]$, with gain group $\mathbb{Z}^+$. We calculate the characteristic polynomial by counting the number of proper colorations of the hollow Catalan gain graph $\{\pm 1\}K_n$ modulo m, i.e., proper group colorings of the gain graph in the group $\mathbb{Z}_m^+$. We have to choose m carefully: no circle can have gain that is a multiple of m. That preserves the balanced chromatic polynomial, by Theorem 87. Since n is the largest possible gain of a circle in $\{\pm 1\}K_n$, the obvious thing to do is to choose $m > n$.

The calculation is similar to that for the Catalan gain graph, but we have to allow for the fact that vertices may have the same color. Thus, we consider a partition of V into k parts, of which there are $S(n,k)$ (the Stirling number of the second kind). The partition consists of the sets of vertices having the same color; each part has one color, and every part has a different color from every other. The number of ways to color the k parts is the same as for the Catalan gain graph $\{0, \pm 1\}K_k$ with k vertices, as in Lecture 13; it is $m(m-k-1)_{k-1}$. We have to multiply this by $S(n,k)$ for the number of k-partitions of V and sum over all possible numbers of parts. We get this:

Proposition 102. *For the hollow Catalan arrangement $\mathscr{C}_n^\circ$ with $n \geq 1$:*

(1) *The characteristic polynomial is*

$$p_{\mathscr{C}_n}(\lambda) = \lambda \sum_{k=1}^{n} S(n,k)(\lambda - k - 1)_{k-1}.$$

(2) *The number of regions is*

$$\sum_{k=1}^{n} S(n,k)(2k)_{k-1} = \sum_{k=1}^{n} S(n,k)k!C_k.$$

(3) *The number of bounded regions is*

$$\sum_{k=1}^{n} S(n,k)(2k-2)_{k-1} = \sum_{k=1}^{n} S(n,k)k!C_{2k-2}.$$

The "finite field" method. The foundation of the finite field method is a theorem of Crapo and Rota. Let's consider an arrangement $\mathscr{A}$ in $\mathbb{A}^n(\mathbb{F}_q)$.

Theorem 103 (Critical Theorem)**.** *The number of points of $\mathbb{A}^n(\mathbb{F}_q)$ not in $\bigcup \mathscr{A}$ is $p_{\mathscr{A}}(q)$.*

Proof. For $x \in \mathscr{L}(\mathscr{A})$, define $f(x) := \#x = q^{\dim(x)}$ and $g(x) := \#(x \setminus \bigcup_{y>x} y)$. Then $f(x) = \sum_{y \geq x} g(y)$, so by Möbius inversion $g(x) = \sum_{y \geq x} f(y)\mu(x,y)$. This equals $\sum_{y \geq x} q^{\dim(y)}\mu(x,y)$. Setting $x = \hat{0}$, we have $\sum_{x \in \mathscr{L}(\mathscr{A})} q^{\dim(y)}\mu(\hat{0},y) = p_{\mathscr{A}}(q)$. □

Now suppose we have an integral arrangement $\mathscr{A}$ in $\mathbb{A}^n(\mathbb{R})$, What is its characteristic polynomial?

Let $\mathscr{A}_p = \mathscr{A} \mod p$ for a prime p, so $\mathscr{A}_p$ is an arrangement in $\mathbb{A}^n(\mathbb{F}_p)$. If $\mathscr{L}(\mathscr{A}_p) \cong \mathscr{L}(\mathscr{A})$, then $p_{\mathscr{A}_p}(\lambda) = p_{\mathscr{A}}(\lambda)$. Now take a prime power, $q = p^e$; we can think of $\mathscr{A}_p$ as generating

an arrangement $\mathscr{A}_q$ (with the same defining equations) in $\mathbb{A}^n(\mathbb{F}_q)$ and we are assured that $\mathscr{L}(\mathscr{A}_p) \cong \mathscr{L}(\mathscr{A}_q)$. There are infinitely many q's for each p, so we could try to calculate $\#(\mathbb{A}^n(\mathbb{F}_{p^e}) \setminus \bigcup \mathscr{A}_{p^e})$ for all $e \geq 1$, therefore getting a formula for $p_{\mathscr{A}}(\lambda)$. (Tip: This is not what people do. But they could.)

The crucial requirement is that $\mathscr{L}(\mathscr{A}_p) \cong \mathscr{L}(\mathscr{A})$. So, when is it true? Let's be more precise about the arrangement. Say $\mathscr{A} = \{h_{\alpha_i,c_i} : i = 1, \ldots, l\}$, with l hyperplanes of the form $h_{\alpha,c} = \{x \in \mathbb{A}^n(\mathbb{R}) : \alpha \cdot x = c\}$.

Since dependence of the hyperplanes corresponds to dependence of the defining equations, look at the matrix $U = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_l \end{pmatrix} \in M_{n\times l}(\mathbb{Z})$ and the $1 \times l$ matrix $c = \begin{pmatrix} c_1 & c_2 & \ldots & c_l \end{pmatrix}$. All the hyperplane equations are represented by the matrix equation $U^T x = c^T$. The solution set of this system of equations is $\bigcap \mathscr{A}$. Now projectivize to $\mathscr{A}_{\mathbb{P}}$ and let $U' := \begin{pmatrix} U \\ c \end{pmatrix}$. For any subarrangement $\mathscr{S} \subseteq \mathscr{A}_{\mathbb{P}}$, the rank is equal to the largest order of a nonsingular square submatrix of $U'_{\mathscr{S}}$, where the subscript means only taking the columns corresponding to hyperplanes in $\mathscr{S}$. If every such submatrix remains nonsingular modulo p, then every subset of columns in U' has the same rank in $U' \mod p$, and that implies $\mathscr{L}(\mathscr{A}_p) \cong \mathscr{L}(\mathscr{A})$. A sufficient condition for preserving nonsingularity is that p does not divide the determinant of any nonsingular square submatrix of U'. It follows that almost all primes, and all sufficiently large primes, give the desired lattice isomorphism. That proves:

Theorem 104 (Finite Field Method)**.** *Given an integral arrangement $\mathscr{A}$ in $\mathbb{A}^n(\mathbb{R})$, for every sufficiently large prime p the modular arrangement $\mathscr{A}_p$ has the same characteristic polynomial as does $\mathscr{A}$.*

Thus, the Critical Theorem enables us to obtain $p_{\mathscr{A}}(\lambda)$ by computing the number of points of $\mathbb{A}^n(\mathbb{F}_p) \setminus \bigcup \mathscr{A}_p$ for all large primes p. This is how the finite field method works. Note that we do not need finite fields, only prime fields. In other words, we may work modulo prime numbers. In fact (but this is not part of the finite field method), we could work modulo any positive integer m that is relatively prime to all the nonzero subdeterminants of U' (no one does this).

Affinographic arrangements. For affinographic arrangements, where every equation has the form $x_j - x_i = c$, the finite-field method is simpler because the matrix U, the top part of U', is totally unimodular (every subdeterminant is 0 or ± 1) so all primes are good as far as concerns determinants in U. The other subdeterminants of U' are those that use c. The only nonzero ones we need to worry about are those associated with a circle C. The circle has l vertices and edges. The $l \times l$ submatrix U_C of U corresponding to those vertices and edges has determinant 0, so it is not of concern, but what is of concern is the $l \times l$ submatrix of U' obtained by substituting for one row (any one row) of U_C the row c_C of c. (In other words take the columns of C with all the vertex rows of C and the gain row c; then delete any one vertex row to get a square matrix.) Call this matrix U'_C; then $\det U'_C = \pm\varphi(C)$. Therefore, in the finite field method we can use any prime that does not divide the gain of an unbalanced circle.

Exercise 105. Prove the preceding paragraph. In particular, prove the determinant formula.

This works. In fact, the determinant formula proves:

Proposition 106. *Given an integral gain graph Φ, the finite field method works on $\mathscr{A}[\Phi]$ for a prime power p^e if and only if p does not divide the gain of any unbalanced circle in Φ.*

But it is simpler to use modular coloring directly. For one thing, we are not restricted to primes. For a second, we immediately see exactly which moduli are valid: every positive integer m that is not a divisor of any nonzero circle gain.

Modular coloring does not appear to count points an affine space, unlike the critical Theorem, but in fact it is not so different. Suppose we have an integral gain graph Φ and consider a coloration $\gamma : V \to \mathbb{Z}_m$. We can view γ as the vector $(\gamma(v_1), \ldots, \gamma(v_n))$ in $\mathbb{Z}_m^V = \mathbb{Z}_m^n$. The rule for γ to be a proper coloration is that it avoids all the hyperplanes of $\mathscr{A}_m[\Phi]$. In other words, the difference is not that great. On the other hand, it is not that little, since the Critical Theorem is false in $\mathbb{Z}_m^n$ if m is composite. It is only the special form of affinographic hyperplanes that lets us use vectors in $\mathbb{Z}_m^n$ (which we call colorations) to get the characteristic polynomial.

Example: The Shi arrangement. This is the arrangement $\mathscr{S}_n = \mathscr{A}[\{0,1\}\vec{K}_n]$ associated with the Shi gain graph, $\{0,1\}\vec{K}_n$. The computation via modular coloring is simple. The 0-edges ensure that no two vertices have the same color, so as with the Catalan arrangement we can put the n vertices into spaces between $m-n$ markers to make a sequence of m places labelled by the colors $0, 1, \ldots, m-1 \in \mathbb{Z}_m$. The rule for the Shi arrangement is that two vertices may have adjacent colors but if they do, say $\gamma(v_i) = \gamma(v_j) \pm 1$ where $i < j$, then $\gamma(v_j) \neq \gamma(v_i) + 1$ due to the 1-edges. That means that if we have a (cyclically) consecutive sequence of colors applied to a bunch of vertices, those vertices must be in decreasing order by subscript. And that means that the order of vertices in a bunch that have consecutive colors is determined. So, all we need to do is place v_1 in the last place of our sequence (position $m-1$) and distribute the other $n-1$ vertices into the $m-n$ spaces arbitrarily (there are $(m-n)^{n-1}$ ways to do that. Then we rotate the sequence so v_1 is in any position (m ways). That gives each proper $\mathbb{Z}_m$ coloration exactly once, so we have the characteristic polynomial.

Proposition 107. *For the Shi arrangement $\mathscr{S}_n$ with $n \geq 1$:*

(1) *The characteristic polynomial is*

$$p_{\mathscr{S}_n}(\lambda) = \lambda(\lambda - n)^{n-1}.$$

(2) *The number of regions is*

$$(n+1)^{n-1}.$$

(3) *The number of bounded regions is*

$$(n-1)^{n-1}.$$

The number of regions equals the number of labelled trees of order $n+1$; this suggests finding an explicit bijection, which has been done (cf. Stanley).

Exercise 108. Does the equivalent of Corollary 97 apply to the Shi arrangement? Think about why there should be a certain answer (yes or no). Then see if you can prove it.

The *fundamental region* of the complete-graph arrangement $\mathscr{A}[0K_n]$ is the region defined by $x_1 < x_2 < \cdots < x_n$. The Catalan arrangement divides this fundamental region into C_n Catalan regions.

Problem 109. Into how many regions does the Shi arrangement divide the fundamental region of the complete-graph arrangement?

Answer using gain graphs. That would be a good research result, especially if you develop a method.

If your method allows, give a complete description of the subregions obtained from the fundamental region of $\mathscr{A}[0K_n]$.

Lecture 15: More of the Same

29 January 2020
Notetaker: Shuchen Mu

The proof of Corollary 97 cannot work for the Shi arrangement $\mathscr{S}_n$ because for each pair of coordinates (i, j) with $i < j$ there is a hyperplane $x_j - x_i = 1$ but no hyperplane $x_i - x_j = 1$. Thus, a transposition does not preserve the Shi arrangement and the image of a Shi region is not a region any more. Nonetheless, the Shi regions have other interesting combinatorics. Since as we have seen

$$p_{\mathscr{S}_n}(\lambda) = \lambda(\lambda - n)^{n-1},$$

thus $r(\mathscr{S}_n]) = (n+1)^{n-1}$, which is well known (since Cayley) to be the number of spanning trees of K_{n+1}. This coincidence naturally invites a combinatorist to seek a proof by bijection—and it has been done.

Example: Root system and threshold arrangements and their perturbations. A main example of two-term hyperplane arrangements $\mathscr{H}[\Phi]$ is arrangements obtained from the classical root systems (already introduced in Lecture 10). Root systems originated in Lie algebra but they have turned out to be widely interesting, including in combinatorics. Stanley [9, Section 5.1] defines the main ones in terms of vectors and dual hyperplanes, but I will define them in terms of vectors and gain graphs over the 2-element gain group $\{\pm\}$, i.e., signed graphs.

There are four infinite sequences of *classical root systems*, written A_{n-1}, B_n, C_n, and D_n, each one naturally described in $\mathbb{R}^n$, and there are also finitely many *exceptional root systems*, which do not fit well with gain graphs so I will ignore them. To describe the classical ones, I write b_i for the ith standard unit vector in $\mathbb{R}^n$. The linear-algebra dual to a vector v is the hyperplane $\{x \in \mathbb{R}^n : v \cdot x = 0\}$.

(1) $A_{n-1} = \{b_i - b_j : i \neq j\}$. The dual arrangement is $\mathscr{A}_{n-1} = \mathscr{H}[+K_n]$, as you can easily verify.
(2) $D_n = A_{n-1} \cup \{\pm(b_i + b_j) : i \neq j\}$. The dual arrangement is $\mathscr{D}_n = \mathscr{H}[\pm K_n]$.
(3) $B_n = D_n \cup \{b_i : i \leq n\}$. The dual arrangement is $\mathscr{B}_n = \mathscr{H}[\pm K_n']$, where the prime denotes a half edge at every vertex. (A half edge has degree 1 and has no gain.)
(4) $C_n = C_n \cup \{2b_i : i \leq n\}$. The dual arrangement is $\mathscr{C}_n = \mathscr{H}[\pm K_n^\circ]$, where the superscript denotes a negative loop at every vertex. While $C_n \neq B_n$, the duals are the same: $\mathscr{B}_n = \mathscr{C}_n$. (Do not confuse this with the Catalan arrangement.)

We met these arrangements in Lecture 10, Example 68 et seq., but at that time I didn't mention the root systems themselves. I add to this list the *threshold arrangement*:

(5) $T_n = \{\pm(b_i + b_j) : i \neq j\}$ and the dual arrangement, which is the threshold arrangement $\mathscr{T}_n = \mathscr{H}[-K_n]$.

There are many questions related to root systems of the following kind:

Problem 110. Generalize a construction or property from affinographic arrangements (like the Catalan and Shi arrangements) to similar affine perturbations of $\mathscr{B}_n$, or possibly $\mathscr{D}_n$. (The main interest for combinatorics at present is in $\mathscr{B}_n$.) For instance, there are Type B Catalan and Shi arrangements. One question, of course, is the characteristic polynomials. Another is to describe the dissection of the fundamental region of the corresponding root

system arrangement, as with the Catalan and Shi arrangements in relation to $\mathscr{A}_{n-1}$. The same questions can be asked for Catalan and Shi threshold arrangements, on which there has been some work: see [5, 6].

Lecture 16: Exponential Sequences

Feb. 3, 2020
Notetaker: Andrew Lamoureux

We consider a remarkable property of certain sequences of arrangements of increasing dimensionality and its interpretation in terms of gain graphs. We start by copying Stanley:

Definition 111 (Stanley's definition)**.** Let K be a field. A sequence $(\mathscr{A}_n \mid n \geq 1)$ of hyperplane arrangements is an *exponential sequence of arrangements* if it satisfies:

(S1) each $\mathscr{A}_n$ is an affine hyperplane arrangement in $\mathbb{A}^n(K)$;
(S2) each $\mathscr{A}_n$ is affinographic (i.e., each hyperplane is an affine translate of a graphic hyperplane); and
(S3) for each n and $B \subseteq [n]$, the hyperplanes with coordinates in B yield an arrangement $\mathscr{A}_n{:}B$ such that $\mathscr{L}(\mathscr{A}_n{:}B) \cong \mathscr{L}(\mathscr{A}_{|B|})$.

I want to impose a stronger axiom than (S3). (A1, A2) are the same but (A3) is more restrictive and implies (S3).

Definition 112 (Our definition)**.** A sequence $(\mathscr{A}_n \mid n \geq 1)$ of hyperplane arrangements is an *exponential sequence of arrangements* if it satisfies:

(A1) each $\mathscr{A}_n$ is an affinographic hyperplane arrangement $\mathscr{A}[\Phi_n]$ in $\mathbb{A}^n(K)$;
(A2) each Φ_n is an K^+-gain graph of order n with $V(\Phi_n) = \{v_i \mid i \in [n]\}$; and
(A3) for each n and $B \subseteq [n]$, the induced subgraph $\Phi_n{:}B \cong \Phi_{|B|}$.

The difference between (S3) and (A3) should not bother us. I expect that in every case of interest, it is the underlying gain graphs, not only the semilattices, that are isomorphic.

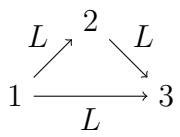
Viewing this in terms of K^+-gain graphs leads to a simple generalization.

Definition 113. Let $\mathfrak{G}$ be a group. A sequence $(\Phi_n \mid n \geq 1)$ of gain graphs is an *exponential sequence of gain graphs* if it satisfies:

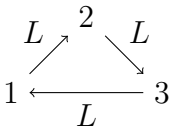
(G1) each Φ_n is a finite $\mathfrak{G}$-gain graph of order n with $V(\Phi_n) = \{v_i \mid i \in [n]\}$; and
(G2) for each n and $B \subseteq V(\Phi_n)]$, the induced subgraph $\Phi_n{:}B \cong \Phi_{|B|}$.

Let's examine (A3) (equivalantly, (G2)) carefully. For $|B| = 1$, it says nothing. For $|B| = 2$, by definition $\Phi_2 = L\vec{K}_2$ for some finite $L \subseteq K$. Hence for all $v_i, v_j \in V(\Phi_n)$, $\Phi_n{:}\{v_i, v_j\} \cong L\vec{K}_2$, and this isomorphism is natural in the sense that it preserves structure (not necessarily in the sense of category theory). Recall that an isomorphism of gain graphs $\alpha : \Phi \to \Phi'$ is an isomorphism $\|\Phi\| \to \|\Phi'\|$ of underlying graphs such that $\varphi(e) = \varphi(e^\alpha)$ for every edge e.

Now consider $|B| = 3$. The gain graph Φ_3 must be one of the following (up to permuting vertices), with the arrows showing the sense in which to read the gains in L:



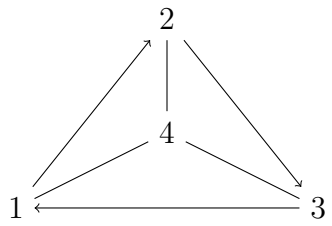
transitive



cyclic

If L is sign-symmetric, i.e., $L = -L$, there is no difference between these two possibilities. Otherwise, there is, for $n \geq 3$: every edge v_iv_j of K_n has a preferred orientation (the one

in which the gain set is L, not $-L$) and every induced subgraph of order 3 of every Φ_n for $n > 3$ is of the same kind: all are transitive, or all cyclic. But if all are cyclic, we have a failure at Φ_4. Suppose the outer triangle 123 in Φ_4 is cyclic, as in the following picture:



To make $\triangle 124$ cyclic, edge 14 must be oriented $1 \to 4$, but to make $\triangle 143$ cyclic, edge 14 must be oriented $4 \to 1$. Hence, Φ_4 cannot be oriented cyclically. The conclusion is:

Proposition 114. *An exponential sequence of gain graphs satisfies $\Phi_n \cong L\vec{K}_n$ for some finite subset $L \subseteq \mathfrak{G}$.*

An exponential sequence of arrangements (per Definition 112) satisfies $\mathscr{A}_n \cong \mathscr{A}[L\vec{K}_n]$ for some finite subset $L \subseteq K^+$.

Proof. Suppose L is not sign-symmetric (otherwise, this is clear). We show that $V(K_n) = \{v_i \mid i \in [n]\}$ can be totally ordered so as to prove the theorem. Define $v_i < v_j$ when the function $v_i \mapsto v_1$, $v_j \mapsto v_2$ is an isomorphism $\Phi_n{:}\{v_i, v_j\} \cong \Phi_2$. This is a strict total order:

(1) (Irreflexivity) There is no bijection $\{v_i\} \to \{v_1, v_2\}$.
(2) (Anti-symmetry) Since $-L \neq L$, it can't be the case that both bijections $\{v_i, v_j\} \to \{v_1, v_2\}$ induce gain-graph isomorphisms; one must give reversed direction of gains.
(3) (Totality) If $v_i \not< v_j$, then the opposite function, $v_i \mapsto v_2$ and $v_j \mapsto v_1$, must be an isomorphism.
(4) (Transitivity) From the analysis of Φ_4 we know that $v_i < v_j < v_k < v_i$ is impossible, so by totality, $v_i < v_j < v_k$ implies $v_k < v_i$.

We have proved that $\Phi_n \cong L\vec{K}_n$ by a permutation of the vertex set, namely, the permutation that carries the ordering $v_{i_1} < v_{i_2} < \cdots < v_{i_n}$ of $V(\vec{K}_n)$ to the natural ordering $v_1 < v_2 < \cdots < v_n$. □

Example 115 (A counterexample)**.** To see that (A3) truly gives a different definition from (S3), consider the following example. Let $\mathfrak{A}$ be an infinite group with an element g of infinite order. Define $\Phi_n := g\vec{K}_n$ for $n \leq 100$ and $\Phi_n := (2g)\vec{K}_n$ for $n > 100$. Then the biased graphs satisfy $\langle \Phi_n \rangle = (K_n, \emptyset)$ for all n, hence $\mathrm{Lat}^{\mathrm{b}}(\Phi_n{:}B) \cong \mathrm{Lat}^{\mathrm{b}}(\Phi_{|B|}$ for every $B \subseteq V(\Phi_n)$, whence $\mathscr{L}(\mathscr{A}_n{:}B) \cong \mathscr{L}(\mathscr{A}_{|B|}$ for every B; yet $\Phi_n{:}\{v_i, v_j\} \neq \Phi_2$ for $n > 100$.

Exercise 116. Prove the statement about the biased graph.

Now we have the surprising main theorem. To simplify the notation I will write $p_n(t)$ for $p_{\mathscr{A}_n}(t)$ and so forth. Recall that $p_n(-1) = (-1)^n r(\mathscr{A}_n)$, the number of regions with a sign; that is how Stanley writes the theorem.

Theorem 117 (Stanley [9, Theorem 5.17])**.** *Let $(\mathscr{A}_1, \mathscr{A}_2, \ldots)$ be an exponential sequence of arrangements. Then*

$$\sum_{n=0}^{\infty} p_n(t)\frac{x^n}{n!} = \Big(\sum_{n=0}^{\infty} p_n(-1)\frac{x^n}{n!}\Big)^{-t}.$$

The expression $\sum_{n=0}^{\infty} p_n(t)\frac{x^n}{n!}$ is the exponential generating function for the sequence $p_n(t)$. Stanley writes $(-1)^n r_n$ instead of $p_n(-1)$ because $r_n := r(\mathscr{A}_n)$ is the number of regions of the arrangement, obviously a number of particular interest. I will adopt that notation in examples in the next lecture. But for now, I wish to rewrite his proof in terms of gain graphs. Thus, I assume we have an exponential sequence of gain graphs Φ_n with balanced chromatic polynomials $\chi^{\mathrm{b}}_n(t) := \chi^{\mathrm{b}}_{\Phi_n}(t)$. We assume that $p_0(t) = \chi^{\mathrm{b}}_0(t) = 1$.

Proof. The theorem can be rewritten as

$$\text{LHS} = \sum_{n\geq 0} \chi^{\mathrm{b}}_n(t)\frac{x^n}{n!} = \Big(\sum_{n\geq 0} \chi^{\mathrm{b}}_n(-1)\frac{x^n}{n!}\Big)^{-t} = \text{RHS}. \tag{5}$$

Our proof takes advantage of the classic *exponential formula*:

$$\sum_{n\geq 0} \frac{x^n}{n!} \sum_{\text{order-}n\text{ objects}O} f(O) = \exp\Big(\sum_{n\geq 1} \frac{x^n}{n!} \sum_{\text{order-}n\text{ connected objects}O} f(O)\Big),$$

provided that f is a function such that $f(O) =$ the product of $f(O')$ over all connected components O' of O. For example, the left side may count the number of n-vertex forests while the right side exponent counts the number of n-vertex trees; the left side may count the number of 2-regular graphs of order n and the right side would count the number of n-vertex circles. (For counting, $f(O) = 1$.) In our case, we are interested in the balanced closed sets in $L\vec{K}_n$ on the left and the connected balanced closed sets in $L\vec{K}_n$ on the right, and instead of counting we are using the balanced chromatic polynomial. Did I mention that for a disjoint union of gain graphs,

$$\chi_{\Phi_1\cup\Phi_2}(\lambda) = \chi_{\Phi_1}(\lambda)\chi_{\Phi_2}(\lambda) \text{ and } \chi^{\mathrm{b}}_{\Phi_1\cup\Phi_2}(\lambda) = \chi^{\mathrm{b}}_{\Phi_1}(\lambda)\chi^{\mathrm{b}}_{\Phi_2}(\lambda)?$$

These formulas follow easily from the definitions and the facts that the size and number of balanced components of an edge set of the disjoint union are additive:

$$|S| = |S\cap E_1| + |S\cap E_2| \text{ and } b(S) = b(S\cap E_1) + b(S\cap E_2).$$

We know that in (5),

$$\text{LHS} = \sum_{n\geq 0} \sum_{S\in \mathrm{Lat}^{\mathrm{b}}\,\Phi_n} \mu(\emptyset, S)t^{n-\mathrm{rk}(S)}\frac{x^n}{n!} = \sum_{n\geq 0} \sum_{S\in \mathrm{Lat}^{\mathrm{b}}\,\Phi_n} \mu(\emptyset, S)t^{b(S)}\frac{x^n}{n!}.$$

Let S have the balanced components $S_1, \dots, S_k$. They yield a partition $\pi(S)$ of $V_n := V(\Phi_n)$ into the subsets $V(S_1), \dots, V(S_n)$. Then $\mu(\emptyset, S) = \prod_{i=1}^k \mu(\emptyset, S_i)$ because the interval from $\emptyset$ to S is a product: $[\emptyset, S] \cong \prod_{i=1}^k [\emptyset, S_i]$ (Exercise!). We can rewrite the Möbius product in terms of $\pi(S)$: $\mu(\emptyset, S)t^{b(S)} = \mu(\emptyset, S)t^k = \prod_{i=1}^k \mu(\emptyset, S_i)t$. Since $S_i = S{:}V(S_i)$ and $\pi(S) = \{V(S_1), \dots, V(S_k)\}$,

$$\begin{aligned}\sum_{S\in \mathrm{Lat}^{\mathrm{b}}\,\Phi_n} \mu(\emptyset, S)t^{b(S)} &= \sum_{S\in \mathrm{Lat}^{\mathrm{b}}\,\Phi_n} \prod_{i=1}^{k} \mu(\emptyset, S_i)t = \sum_{S} \prod_{B\in\pi(S)} \mu(\emptyset, S{:}B)t \\ &= \sum_{\pi\in\Pi_n} \prod_{B\in\pi} \Big(\sum_{S:\,\pi(S)=\pi} \mu(\emptyset, S{:}B)t\Big) = \sum_{\pi} \prod_{B\in\pi} \tilde{\chi}_{|B|}(t),\end{aligned}$$

where we define the convenient notation

$$\tilde{\chi}_n(t) := \sum_{S \in \mathrm{Lat}^{\mathrm{b}}\, \Phi_n \colon \pi(S) = \{[n]\}} \mu(\emptyset, S) t^{b(S)} = \sum_{S \in \mathrm{Lat}^{\mathrm{b}}\, \Phi_n \colon \pi(S) = \{[n]\}} \mu(\emptyset, S) t$$

because $b(S) = 1$ if $\pi(S) = \{[n]\}$.

Now we rewrite the left side in (5) using magic:

$$\begin{aligned} \mathrm{LHS} &= \sum_{n \geq 0} \chi^{\mathrm{b}}_n(t) \frac{x^n}{n!} = \sum_{n \geq 0} \sum_{\pi} \prod_{B \in \pi} \tilde{\chi}_{|B|}(t) t^{b(S)} \frac{x^n}{n!} = \exp\Big(\sum_{n \geq 1} \tilde{\chi}_n(t) \frac{x^n}{n!} \Big) \\ &= \exp\Big(t \sum_{n \geq 1} \sum_{S \in \mathrm{Lat}^{\mathrm{b}}\, \Phi_n \colon \pi(S) = \{[n]\}} \mu(\emptyset, S) \frac{x^n}{n!} \Big) \\ &= \Big[\exp\Big(\sum_{n \geq 1} \sum_{S \in \mathrm{Lat}^{\mathrm{b}}\, \Phi_n \colon \pi(S) = \{[n]\}} \mu(\emptyset, S) \frac{x^n}{n!} \Big) \Big]^t. \end{aligned}$$

We substitute $t = -1$ to get another formula:

$$\sum_{n \geq 0} \chi^{\mathrm{b}}_n(-1) \frac{x^n}{n!} = \exp\Big(\sum_{n \geq 1} \tilde{\chi}_n(-1) \frac{x^n}{n!} \Big) = \Big[\exp\Big(\sum_{n \geq 1} \sum_{S \in \mathrm{Lat}^{\mathrm{b}}\, \Phi_n \colon \pi(S) = \{[n]\}} \mu(\emptyset, S) \frac{x^n}{n!} \Big) \Big]^{-1}.$$

Compare the last expressions of these two formulas: They are the same except for the exponent. Therefore,

$$\Big(\sum_{n \geq 0} \chi^{\mathrm{b}}_n(-1) \frac{x^n}{n!} \Big)^{-t} = \mathrm{LHS},$$

which proves (5) and the theorem. □

The proof never uses arrangements; it is valid for any exponential sequence of gain graphs. That is, we have a generalization independent of fields.

Theorem 118. *If $(\Phi_n \mid n \geq 0)$ is an exponential sequence of gain graphs, then*

$$\sum_{n=0}^{\infty} \chi^{\mathrm{b}}_n(t) \frac{x^n}{n!} = \Big(\sum_{n=0}^{\infty} \chi^{\mathrm{b}}_n(-1) \frac{x^n}{n!} \Big)^{-t}.$$

(Stanley's Exercise 5.10 is a much more interesting and less expected generalization.)

Example 119. Suppose $\mathfrak{G}$ is a finite group and $\Phi_n = \mathfrak{G}K_n$. Then Theorem 118 applies. In this case we know the balanced chromatic polynomial: it is $|\mathfrak{G}|^n (t/|\mathfrak{G}|)_n$ and the evaluation at -1 is $(|\mathfrak{G}| - n + 1)(|\mathfrak{G}| - 2n + 1) \cdots (1)$.

Lecture 17: More Exponential Sequences; More Gain Expansions

5 February 2020
Notetaker: Mike Gottstein

Let's explore some applications of Theorem 117 (Stanley's Theorem 5.17).

Example 120 (Catalan as exponential sequence)**.** We begin with the Catalan arrangement $\mathscr{C}_n = \mathscr{A}[\{0, \pm 1\}K_n]$ and what we can do for it with Theorem 117. The Catalan generating function has the following well known formula (e.g., see Wikipedia!):

$$C(x) := \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x} = \frac{2}{1+\sqrt{1+4x}}. \tag{6}$$

Using gain graphs we formulated the following balanced chromatic polynomial for the affinographic arrangement $\mathscr{A}[\{\pm 1, 0\}K_n]$, which is the Catalan arrangement $\mathscr{C}_n$. $\chi^{\mathrm{b}}_{\{\pm 1,0\}K_n}(\lambda)$ which we have shown is equal to the characteristic polynomial $p_{\mathscr{C}_n}(\lambda)$. The Catalan sequence $(\mathscr{C}_1, \mathscr{C}_2, \ldots)$ is an ESA, so by Theorem 117, $r(\mathscr{C}_n) = n!C_n$, and Equation (6),

$$\begin{aligned}\sum_{n\geq 0} p_{\mathscr{C}_n}(t)\frac{x^n}{n!} &= \Big(\sum_{n\geq 0}(-1)^n n! C_n \frac{x^n}{n!}\Big)^{-t} = \Big(\sum_{n\geq 0} C_n(-x)^n\Big)^{-t} \\ &= \Big(\frac{1-\sqrt{1+4x}}{2x}\Big)^{-t} = \Big(\frac{1+\sqrt{1-4x}}{2}\Big)^{t}.\end{aligned}$$

Substituting $t = +1$, the left-hand side is $\sum_{n\geq 0}(-1)^n b(\mathscr{C}_n)\frac{x^n}{n!}$, where $b(\mathscr{C}_n)$ is the number of bounded regions of $\mathscr{C}_n$, so

$$\sum_{n\geq 0} b(\mathscr{C}_n)\frac{(-x)^n}{n!} = \frac{1+\sqrt{1-4x}}{2}.$$

Example 121 (The complete graph as exponential sequence)**.** Let's see Theorem 117 at work in an obvious case. Consider the complete graph arrangements $\mathscr{A}[K_n]$, where $p_n(t) = (t)_n$ and $r_n = n!$. In Theorem 117 the left side is $\sum_{n\geq 0}\binom{t}{n}x^n = (1+x)^t$ by the binomial series, while the right hand side is $\big(\sum_{n\geq 0} n!\frac{(-x)^n}{n!}\big)^{-t}$, which equals $(\frac{1}{1+x})^{-t} = (1+x)^t$ by the geometric series. So the theorem holds here, unsurprisingly.

Example 122 (Shi as exponential sequence)**.** Now a not-so-obvious case, the Shi arrangements $\mathscr{S}_n$. Here $p_n(t) = t(t-n)^{n-1}$ and $r_n = (n+1)^{n-1}$, so by Theorem 117,

$$\sum_{n\geq 0} t(t-n)^{n-1}\frac{x^n}{n!} = \Big(\sum_{n\geq 0}(n+1)^{n-1}\frac{-x^n}{n!}\Big)^{-t},$$

which is (to put it mildly) a less trivial identity to check.

Arrangements connected to interval orders. An interval order is a partially ordered set that can be represented by intervals $I_i = [a_i, b_i]$ for $i = 1, 2, \ldots, t$ in the real line, with $I_i < I_j \iff b_i < a_j$. See Stanley [9, Section 5.5] for more about interval orders. Here my interest is in the arrangements he finds in that connection.

The arrangements are of the following kind: Take a finite subset $L = \{l_1, l_2, \ldots, l_t\} \subset \mathbb{R}_{>0}$. Stanley's arrangement $\mathscr{I}_L$ in $\mathbb{A}^n(\mathbb{R})$ has the hyperplanes $x_j - x_i = l_k$ for all $i \neq j$ and all $k = 1, \ldots, t$. This is precisely the affinographic arrangement $\mathscr{A}[\pm LK_n]$.

Holding L fixed, these arrangements, or gain graphs, form an exponential sequence; but we would have trouble applying Theorem 117 because we have no way to compute the characteristic polynomial (that is, the balanced chromatic polynomial $\chi^{\mathrm{b}}_{\pm LK_n}(\lambda)$) or the number of regions.

The gain graph $\pm LK_n$ is a kind of gain expansion of K_n, similar to group expansions (Theorem 63) but not so easy to compute with since we expand by a small subset of the gain group $\mathbb{R}^+$. We have no general formula for $\chi^{\mathrm{b}}_{\pm LK_n}(\lambda)$. However, if we can convert the gains l_i to integers, we would have gain group $\mathbb{Z}^+$ and we could apply modular coloring. But is that possible? Yes!

Proposition 123 (Special Integralization)**.** *Suppose $L = \{l_1, l_2, \ldots, l_n\} \subset \mathbb{R}_{>0}$. Then there exists $L' = \{l_1, l_2, \ldots, l_n\} \subset \mathbb{Z}_{>0}$ with the same biased graph: $\langle \pm LK_n \rangle = \langle \pm L'K_n \rangle$, hence the gain graphs have the same balanced chromatic polynomial.*

The proof will appear in the next lecture.

The original numbers l_i are real numbers because they are the lengths of intervals in the real line. By converting them to integers we make it possible, in principle at least, to compute the balanced chromatic polynomial by modular coloring. In practice each example would have to be handled separately. That suggests a completely open problem.

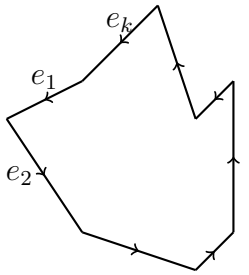
Problem 124. Find a modular coloring solution for a whole class of sets L. Any class—except the class with $t = 1$, for which $\pm LK_n$ is essentially the hollow Catalan gain graph (their biased graphs and polynomials are the same).

Lecture 18: Additive Real Gain Graphs: Integralization

14 February 2020
Notetaker: Nicholas Lacasse

Proof of Proposition 123. Let $L = \{l_1, \ldots, l_t\} \subset \mathbb{R}^+_{>0}$ where $\mathbb{R}^+_{>0}$ denotes the additive semigroup of positive real numbers, and consider $\pm LK_n$, the gain graph with gain group $\mathbb{R}^+$, and its affinographic arrangement $\mathscr{A}[\pm LK_n]$. The objective is to find a set of integral gains, $L' = \{l'_1, \ldots, l'_t\} \subset \mathbb{Z}^+_{>0}$, such that $\langle L'K_n \rangle = \langle LK_n \rangle$, because to guarantee that we obtain the same balanced chromatic polynomial, we must keep the same set of balanced circles. How do we achieve this?

Consider a circle C as in the figure below, where each $\varphi(e_j) =$ some $\pm l_{i_j}$.



Here $\varphi(C) = \varepsilon_1 l_{i_1} + \varepsilon_2 l_{i_2} + \cdots + \varepsilon_k l_{i_k}$ where $\varepsilon_j \in \{-1, 1\}$ and depends on $\varphi(e_j)$. If C is balanced, then $\varphi(C) = 0$, and if C is not balanced, then $\varphi(C) \neq 0$. We need to choose L' so that it preserves balance and imbalance, i.e., $\varphi'(C) = 0$ if and only if $\varphi(C) = 0$. So we have an equation or inequation of the form

$$\varepsilon_1 l_{i_1} + \varepsilon_2 l_{i_2} + \cdots + \varepsilon_k l_{i_k} \begin{cases} = 0, \\ \neq 0. \end{cases}$$

That is almost enough to get the set L, but we also need to have t distinct values, none equal to 0. Thus, we also need to state that $l_i \neq 0$ and $l_i \neq \pm l_j$ for $i \neq j$. There is a simplification: since we use gains $\pm l_i$, it does not matter whether l_i is positive or negative.

Now we replace the t specific values l_i with t variables $x_1, \ldots, x_t$ to obtain, for each circle, an equation $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_k l_{i_k} = 0$ or an inequation $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_k l_{i_k} \neq 0$. We have one equation per balanced circle, one inequation per unblaanced circle, and t^2 inequations $x_i \neq 0$, $x_i \neq x_j$, and $x_i \neq -x_j$ for $i \neq j$.

This gives us a system of equalities and inequalities. We look for an integer solution using linear algebra. That solution is guaranteed by Theorem 125. □

Theorem 125 (Integralization)**.** *Given $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \in \mathbb{Q}^t$, the requirements that all $\alpha_i \cdot x = 0$ and all $\beta_j \cdot x \neq 0$, and the existence of a real solution $y \in \mathbb{R}^t$, then there exists a rational solution in $\mathbb{Q}^t$.*

After obtaining a rational solution, since t is finite, we can scale by an appropriate integer to obtain an integral solution.

Proof. Let the matrix A have rows $a_i^\perp$. Consider the equation $Ax = 0 \in \mathbb{R}^p$, where $x \in \mathbb{R}^t$. A dot product equation $\alpha \cdot x = 0$ for $\alpha \in \mathbb{R}^t$ is forced to be true if and only if $\alpha \in \mathrm{Row}(A)$. Therefore, because y exists, none of the β_j's are in $\mathrm{Row}(A)$. We assume henceforth, by

discarding unnecessary rows, that A has full row rank r; that simplifies notation in the next step and it is permissible because we retain the same row space.

The solution space of $Ax = 0$ is $\operatorname{Nul} A$, which can be given in terms of parameters, say $x_{r+1}, \ldots, x_t$. Since A is an $r \times t$ matrix, its reduced row echelon form is $\begin{pmatrix} I_r \mid A' \end{pmatrix}$ and the condition for its null space becomes $\begin{pmatrix} I_r \mid A' \end{pmatrix} \begin{pmatrix} \hat{x} \\ \bar{x} \end{pmatrix} = 0$, where $\hat{x} = (x_1, \ldots, x_r)$ and $\bar{x} = (x_{r+1}, \ldots, x_t)$, the parameter vector. So $I_r \hat{x} + A' \bar{x} = 0$ and consequently $\hat{x} = -A' \bar{x}$. Let $B = -A'$. Then

$$\operatorname{Nul} A = \left\{ \begin{pmatrix} I \\ B \end{pmatrix} \bar{x} : \bar{x} \in \mathbb{R}^{t-r} \right\} \subseteq \mathbb{R}^t.$$

So our solution is $\{ \begin{pmatrix} \bar{x} \\ B\bar{x} \end{pmatrix} : \bar{x} \in \mathbb{R}^{t-r} \}$. Because A was a rational matrix and B was obtained from A by row operations, B is also a rational matrix. Therefore $\bar{x} \in \mathbb{Q}^{t-r}$ implies $x \in \mathbb{Q}^t$. Therefore we have found a rational solution. We can choose $\bar{x}$ arbitrarily near $\bar{y}$ so that x is arbitrarily near y. Then each $\beta_j \cdot x$ is changed too little to become 0, therefore we preserve all inequations $\beta_j \cdot x \neq 0$. □

Now we know we can replace real gains by rational gains and therefore by integral gains. This raises a natural question: How much (or little) do we need to perturb the real gains to obtain the rational gains? For instance, what is the smallest integer $d > 0$ such that perturbing y by $< 1/d$ gives a rational solution? Put differently, what is the smallest D such that rounding Dy to the nearest integer vector gives a solution? More simply, we might try multiplying the real gains by 10^m (for some positive integer m) and then rounding to the nearest integer. But what m is sufficiently large?

Virtually the same proof as that of Proposition 123 works for any additive real gain graph. Thus:

Theorem 126 (Gain Graph Integralization)**.** *Let Φ be any $\mathbb{R}^+$-gain graph. Then there exists a $\mathbb{Z}^+$-gain graph Φ' with the same biased graph and therefore the same chromatic polynomials.*

We can infer even more extensive conclusions from Theorem 126. Every complex additive gain graph Φ can be replaced by a gain graph Φ' whose gains are Gaussian integers. Indeed, gains in any real vector space $\mathbb{R}^d$ can be replaced by vectors in $\mathbb{Z}^d$. The original gains could even be polynomials over $\mathbb{R}$ or $\mathbb{C}$, since we never multiply them. There is no problem with infinite dimensionality because our edge set is finite so the gains span a finite-dimensional vector space.

Notice that $\pm LK_n$ is similar to a hollow extended Catalan arrangement, which by definition is $\mathscr{A}[\pm[1,t]_{\mathbb{Z}} K_n]$ where $\pm[1,t]_{\mathbb{Z}}$ is the interval of integers from 1 to t. I will develop this thought in the final lecture.

Lecture 19: Additive Real Gain Graphs: Genericity and Biased Union

17 February 2020
Notetaker: Shuchen Mu

In [9, Section 5.5] Stanley introduces "generic" exponential sequences $\mathscr{A}[\pm LK_n]$, $n \geq 0$, where $L = \{l_1, \ldots, l_t\} \subset \mathbb{R}^+_{>0}$. He gives two proposed definitions of genericity.

(S1) $\mathcal{L}(\mathscr{A}[\pm LK_n])$ [that is, $\mathrm{Lat}^{\mathrm{b}}(\pm LK_n)$] is as big as possible.

(S2) The l_i's are linearly independent over $\mathbb{Q}$.

Does (S1) means $\pm LK_n$ has the fewest possible balanced circles? That suggesets another definition based on thinking about gain graphs:

(T1) L is *generic* if, for all n, $\mathscr{B}(\pm LK_n)$ is as small as possible. Restated,

$$\mathscr{B}(\pm LK_n) = \bigcup_i \mathscr{B}(\pm l_i K_n),$$

since we necessarily have the balanced circles of $\pm K_n$.

(T1) is slightly different from (S2), since we only ask that no circle with different l_i's in it can give gain 0 for any n, which is implied by rational independence of the l_i. However, it is easy to prove that because we have an exponential sequence they are equivalent.

Proposition 127. (T1) $\iff$ (S2).

Proof. Exercise. □

Now we examine the concept of a "bigger" semilattice or lattice of a biased graph.

Proposal 128. Given a graph Γ and two linear classes of circles, $\mathscr{B}_1 \subset \mathscr{B}_2$, then $\mathrm{Lat}^{\mathrm{b}}(\Gamma, \mathscr{B}_1)$ is bigger than $\mathrm{Lat}^{\mathrm{b}}(\Gamma, \mathscr{B}_2)$.

"Bigger" is intentionally not defined, but it should mean something like existence of an order-preserving, or order- and rank-preserving, injective function $\mathrm{Lat}^{\mathrm{b}}(\Gamma, \mathscr{B}_2) \to \mathrm{Lat}^{\mathrm{b}}(\Gamma, \mathscr{B}_1)$ that is not surjective. Or, it may mean the existence of an order-preserving surjective function $\mathrm{Lat}^{\mathrm{b}}(\Gamma, \mathscr{B}_1) \to \mathrm{Lat}^{\mathrm{b}}(\Gamma, \mathscr{B}_2)$ that is not injective. The latter definition leads to a proof of Proposal 128, in Corollary 131.

Exercise 129. Decide whether both definitions of "bigger" are equivalent. Hint: It may easier if you generalize to finite posets.

We get a better understanding of "bigger" from the following property of the balanced-flat semilattice.

Theorem 130. *Given a loopless graph Γ and two linear classes of circles, $\mathscr{B}_1 \subset \mathscr{B}_2$, then there is an order-preserving surjective mapping $\mathrm{Lat}^{\mathrm{b}}(\Gamma, \mathscr{B}_1) \to \mathrm{Lat}^{\mathrm{b}}(\Gamma, \mathscr{B}_2)$ that is not injective.*

Proof. Write $\Omega_1 = (\Gamma, \mathscr{B}_1)$ and $\Omega_2 = (\Gamma, \mathscr{B}_2)$. A mapping $\beta : \mathrm{Lat}\,\Omega_1 \to \mathrm{Lat}\,\Omega_2$ is defined by $\beta(S) = \mathrm{cl}_2(S)$, where cl_i is the closure in Ω_i. (Notice that we defined β on all edge sets.) We have to prove that β is order-preserving and surjective. It is obvious that it maps $\mathrm{Lat}\,\Omega_1 \to \mathrm{Lat}\,\Omega_2$. It maps balanced flats to balanced flats because, by the hypothesis, a balanced set of Ω_1 is also balanced in Ω_2.

Thus, the closure of S in both biases is its balance-closure, defined by

$$\begin{aligned}\operatorname{bcl}_1(S) &= S \cup \{e \notin S : \exists\ C \in \mathscr{B}_1,\ C \cup e \in \mathscr{B}_1\} \\ &\subseteq S \cup \{e \notin S : \exists\ C \in \mathscr{B}_2,\ C \cup e \in \mathscr{B}_1\} = \operatorname{bcl}_2(S).\end{aligned}$$

It follows that $\operatorname{cl}_2(S) \supseteq \operatorname{cl}_1(S)$ for any balanced edge set.

Clearly, β preserves set containment, that is, lattice order. We must prove β is surjective. For $A \in \operatorname{Lat}^{\mathrm{b}} \Omega_2$, choose an Ω_2-basis B of A. It is balanced and independent in Ω_2, hence it is a forest, so it is balanced in Ω_1. Thus, $\beta(B) = \operatorname{cl}_1(B)$ is balanced. Now,

$$B \subseteq \operatorname{cl}_1(B) \subseteq \operatorname{cl}_2(B) = A$$

so

$$A = \operatorname{cl}_2(B) \subseteq \operatorname{cl}_2(\operatorname{cl}_1(B) \subseteq \operatorname{cl}_2(A) = A.$$

Therefore, $\beta(\operatorname{cl}_1(B)) = A$. This proves β is surjective from $\operatorname{Lat}^{\mathrm{b}} \Omega_1$ to $\operatorname{Lat}^{\mathrm{b}} \Omega_2$.

To prove $\beta : \operatorname{Lat}^{\mathrm{b}} \Omega_1 \to \operatorname{Lat}^{\mathrm{b}} \Omega_2$ is not injective, choose a circle $C \in \mathscr{B}_2 \setminus \mathscr{B}_1$. For $e \in C$, $e \notin \operatorname{cl}_1(C \setminus e)$ but $e \in \operatorname{cl}_2(C \setminus e)$ so $\operatorname{cl}_2(C \setminus e) = \operatorname{cl}_2(C)$. The same applies to another edge $f \in C$, which exists because there are no loops (one-edge circles). Now, $\operatorname{cl}_1(C \setminus e)$ and $\operatorname{cl}_1(C \setminus f)$ are two balanced flats in Ω_1 with the same image, $\operatorname{cl}_2(C)$, under β. □

Corollary 131. (S1) $\iff$ (T1).

Proof. Apply Theorem 130, since $\mathscr{L}(\mathscr{A}[\pm LK_n]) \cong \operatorname{Lat}^{\mathrm{b}}(\pm LK_n)$. □

In other words, we have proved that Stanley's two definitions are equivalent (especially if the answer in Exercise 129 is positive—hint: it is).

I purposely omitted proposing that $\operatorname{Lat}(\Gamma, \mathscr{B}_1)$ is bigger than $\operatorname{Lat}(\Gamma, \mathscr{B}_2)$.

Problem 132. Does there necessarily exist an order-preserving surjection $\operatorname{Lat}(\Gamma, \mathscr{B}_1) \to \operatorname{Lat}(\Gamma, \mathscr{B}_2)$?

Although Exercise 129 would imply an injection $\operatorname{Lat}^{\mathrm{b}} \Omega_2 \to \operatorname{Lat}^{\mathrm{b}} \Omega_1$ exists, it would be more valuable to present one explicitly,

Problem 133. Find an explicit formula for such an order-preserving injection, preferably one that is reasonably simple, if that is possible.

Biased union. It is time to introduce a new way of combining biased graphs. The *biased union* of $\Omega_1 = (\Gamma_1, \mathscr{B}_1)$ and $\Omega_2 = (\Gamma_2, \mathscr{B}_2)$, whose edge sets are disjoint, is

$$\Omega_1 \sqcup \Omega_2 = (\Gamma_1 \cup \Gamma_2, \mathscr{B}_1 \cup \mathscr{B}_2).$$

Theorem 134. *The biased union is a biased graph.*

Proof. Exercise. □

Example 135. To illustrate balance and closure in a biased union, see Figures 13–17.

The n^{th} gain graph of our exponential sequence, $\pm LK_n$, contains $\pm l_i K_n$ for each i (which is isomorphic to the hollow Catalan gain graph). Definition (T1) states that L is generic if $\pm LK_n$ is the biased union

$$\langle \pm LK_n \rangle = \bigsqcup_{i=1}^{t} \langle \pm l_i K_n \rangle.$$

Now, here is the crucial question and the purpose of introducing the biased union. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Omega = \Omega_1 \sqcup \Omega_2$.

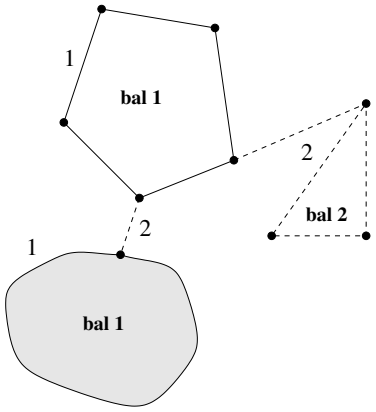


FIGURE 13. A balanced set S in the biased union Ω. The edges belonging to Ω_1 are solid and those belonging to Ω_2 are dashed; and they are labelled. Here there are five blocks, each one balanced in one of Ω_1 and Ω_2. Two blocks are isthmi of S; two are circles; the bottom one is more than a circle.

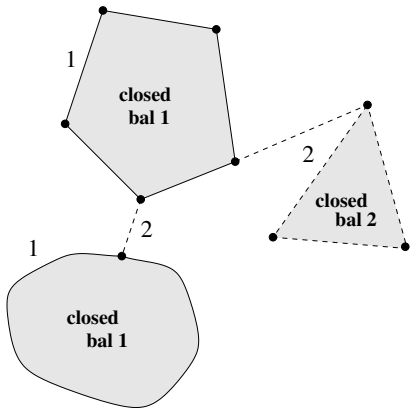


FIGURE 14. A balanced flat in Ω. Each block is balanced and closed in one of Ω_1 and Ω_2.

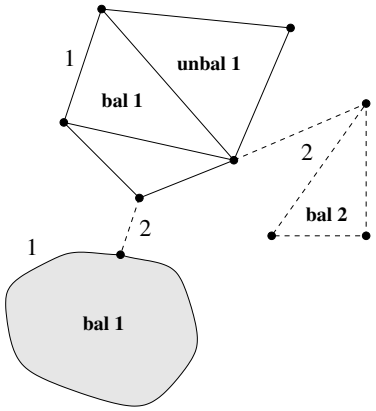


FIGURE 15. An unbalanced set S in Ω. The upper block is contained in Ω_1 but it is unbalanced.

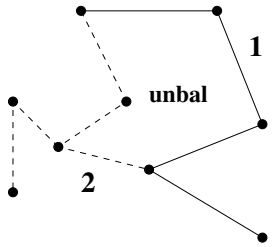


FIGURE 16. An unbalanced set. The polygon is unbalanced because it mixes edges from Ω_1 and Ω_2.

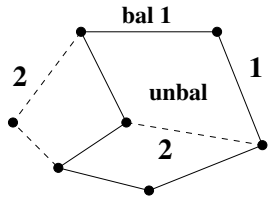


FIGURE 17. An unbalanced set S. The polygon from Ω_1 is balanced but S is a block and it also contains edges from Ω_2 forming unbalanced circles.

Problem 136. Can we express $\chi^{\mathrm{b}}_{\Omega}(\lambda)$ in terms of $\chi^{\mathrm{b}}_{\Omega_1}(\lambda)$ and $\chi^{\mathrm{b}}_{\Omega_2}(\lambda)$ and possibly other information that we already know from Ω_1 and Ω_2?

Can we similarly infer $\mathrm{Lat}^{\mathrm{b}}\,\Omega$?

Recall the formulas:

$$\chi^{\mathrm{b}}_{\Omega}(\lambda) = \sum_{\substack{S\subseteq E\\ \text{balanced}}} (-1)^{|S|}\lambda^{c(S)} = \sum_{A\in \mathrm{Lat}^{\mathrm{b}}\,\Omega} \mu(\emptyset, A)\lambda^{c(A)}.$$

I would like to somehow use these formulas to extrapolate $\chi^{\mathrm{b}}_{\Omega}$ from the sets that are balanced and closed in the two Ω_i. For instance, (assuming no loops or balanced digons) some closed sets are $\emptyset$ and $\{e\}$ for every edge of the union. But it gets complicated. For instance, every forest of the union Γ is balanced, even if it combines edges of both Ω_i, but not necessarily closed. Now, consider a subset $S \subseteq E(\Gamma)$ (e.g., as in Figure 14): it is balanced and closed if and only if every block is balanced and closed. Suppose, then, that S is a block: it is balanced and closed if and only if it is a closed, balanced, inseparable subset in Ω_1 or Ω_2. So, a balanced flat S of Ω is assembled from inseparable balanced flats of the Ω_i that are disjoint or attached at single vertices so that each is a block of S. Does that give us enough insight to compute the balanced chromatic polynomial of Ω, or even the semilattice of balanced flats?

In the special case of an exponential sequence $\langle \pm LK_n\rangle$ for generic L, perhaps some version of the exponential formula might be able to give a solution. That is the motivation for this discussion.

Here is a thought about generalization. It is surely too hard to solve in general, as even the special case of biased union is unclear.

Problem 137. Suppose we define

$$\Omega_1 \cup \Omega_2 = (V_1 \cup V_2, E_1 \cup E_2, \mathscr{B})$$

where $\mathscr{B}$ is the smallest linear class such that $\mathscr{B} \supseteq \mathscr{B}_1 \cup \mathscr{B}_2$. What is $\mathscr{B}$? What are the properties? Can we describe $\operatorname{Lat} \Omega_1 \cup \Omega_2$ or $\operatorname{Lat}^{\mathrm{b}} \Omega_1 \cup \Omega_2$ in terms of Ω_1 and Ω_2?

The principal question here is whether $\mathscr{B}$ has an explicit description. Only then can any more be thought about.

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